FORELESNING 18

MET 1180 BI

MATEMATIKK

EIVIND ERIKSEN

03 FEB 2016

Plan:

1. Repetisjon: Delvis integrasjon
2. Delbrøksoppspalting
3. Bestemte integral og areal

Pensum:

[S] 9.1-9.2, 9.4

[E] 5.5-5.6

1 Repetisjon: Delvis integrasjon

$$\boxed{\begin{aligned} \int u'v\,dx &= uv - \int uv'dx \\ \int uv'\,dx &= uv - \int u'v\,dx \end{aligned}}$$

Ex:

$$\int (x+1)\cdot \ln x \; dx$$

$$\boxed{\begin{aligned} u &= \tfrac{1}{2}(x+1)^2 & v &= \ln x \\ u' &= x+1 & v' &= \tfrac{1}{x} \end{aligned}}$$

$$= \frac{1}{2}(x+1)^2\cdot \ln x - \int \frac{1}{2}(x+1)^2\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{2}(x+1)^2 \ln x - \frac{1}{2}\int \frac{(x+1)^2}{x}\,dx = \frac{1}{2}(x+1)^2\ln x - \frac{1}{2}\int \frac{x^2+2x+1}{x}\,dx$$

$$\frac{x^2+2x+1}{x} = x+2+\frac{1}{x}$$

$$= \frac{1}{2}(x+1)^2\ln x - \frac{1}{2}\left(\frac{1}{2}x^2 + 2x + \ln x\right) + C$$

$$= \frac{1}{2}(x+1)^2 \ln x - \frac{1}{4}x^2 - x - \frac{1}{2}\ln x + C$$

(2) Delbrøks oppspaltning:

BI

Ex: $\int \frac{x-1}{x^2+5x+6}\,dx$ ← polynomdivisjon? (bruk polynomdiv. først om teller har minst like stor grad som nevner)

Nevner: $x^2+5x+6 = (x+2)(x+3)$ ← hvis nevner har grad 1: substitusjon u = nevner

$x^2+5x+6=0$

$x = \frac{-5 \pm \sqrt{25-4\cdot 6}}{2} = \frac{-5\pm 1}{2}$

$x = -3, -2$

A, B er tall.

$$\frac{x-1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \quad \Big| \cdot (x+2)(x+3)$$

$$x-1 = A\cdot(x+3) + B\cdot(x+2)$$

Alt. 1.	Alt 2.
$x-1 = Ax+3A+Bx+2B$	$x-1 = A(x+3)+B(x+2)$
$x-1 = \underbrace{(A+B)}_{1}x + \underbrace{(3A+2B)}_{-1}$	$x=-3$: $-4 = B\cdot(-1) \Rightarrow B=4$
$A+B=1$, $3A+2B=-1$	$x=-2$: $-3 = A\cdot 1 \Rightarrow A=-3$
$B=1-A$, $3A+2(1-A)=-1$, $A=-1-2$, $A=-3$, $B=4$	Sjekk: $\frac{-3}{x+2}+\frac{4}{x+3} = \frac{-3\cdot(x+3)}{(x+2)(x+3)} + \frac{4(x+2)}{(x+2)(x+3)}$
$\frac{x-1}{(x+2)(x+3)} = \frac{-3}{x+2}+\frac{4}{x+3}$	$= \frac{-3x-9+4x+8}{(x+2)(x+3)} = \frac{x-1}{(x+2)(x+3)}$

$$\int \frac{x-1}{x^2+5x+6}\,dx = \int \left(\frac{-3}{x+2} + \frac{4}{x+3} \right) dx$$

delbrøks-
oppspalting

$$\boxed{\begin{aligned} u &= x+2 \\ du &= 1 \cdot dx \end{aligned}}$$

$$\int \frac{-3}{u}\,du = -3 \cdot \ln|u| + C$$
$$= -3 \ln|x+2| + C$$

$$= -3 \cdot \ln|x+2| + 4 \cdot \ln|x+3| + C$$

Ex:

$$\int \frac{x}{x^2+6x+9}\,dx$$

$$x^2+6x+9 = (x+3)^2$$

$$\frac{x}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

$$x = A\cdot(x+3) + B$$
$$= \underbrace{A}_{=1} \cdot x + \underbrace{(3A+B)}_{=0}$$

$A = 1 \quad B = -3$

$$= \int \left(\frac{1}{x+3} + \frac{-3}{(x+3)^2} \right) dx$$

$$= \ln|x+3| - 3 \int (x+3)^{-2}\,dx$$

$$= \ln|x+3| - 3 \cdot \int u^{-2}\,du$$

$$\boxed{\begin{aligned} u &= x+3 \\ du &= 1 \cdot dx \end{aligned}}$$

$$= \ln|x+3| - 3 \cdot \frac{u^{-1}}{-1} + C = \ln|x+3| + 3 \cdot \frac{1}{u} + C$$

$$= \ln|x+3| + \frac{3}{x+3} + C$$

Eks: $\displaystyle\int \frac{x}{x^2+1}\,dx$

$x^2+1=0$
ingen løsn
⇓
ingen fakt. i lin. faktorer

$\boxed{\begin{array}{l} u=x^2+1 \\ du=2x\cdot dx \end{array}}$

$$= \int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}} = \frac{1}{2}\int\frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1)+C$$

Eks: $\displaystyle\int \frac{1}{x^2+1}\,dx = \int \frac{1}{u}\cdot\frac{du}{2x} = \,?$

$\boxed{\begin{array}{l} u=x^2+1 \\ du=2x\,dx \end{array}}$

$$\boxed{\textbf{Formel:}\quad \int \frac{1}{x^2+1}\,dx = \arctan(x)+C}$$

Trigonometriske funksjoner

Ukke pensum: $\sin x, \cos x$, etc.

$\tan x = \dfrac{\sin x}{\cos x}$

$\arctan x = (\tan x)^{-1}$ omvendt funksjon

Ex: $\displaystyle\int \frac{3x+2}{x^2+1}\,dx =$

$\boxed{\begin{array}{l} u=x^2+1 \\ du=2x\,dx \end{array}}$

$$\int \frac{\frac{3}{2}(2x)+2}{x^2+1}\,dx = \int \frac{\frac{3}{2}\cdot 2x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= \int \frac{\frac{3}{2}\cdot\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}} + \int \frac{2}{x^2+1}\,dx$$

$$= \frac{3}{2}\ln|u| + 2\cdot\arctan(x) + C$$

$$= \frac{3}{2}\ln(x^2+1) + 2\arctan(x) + C$$

③ Bestemte integral og areal

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

der $F(x)$ er en antiderivert til $f(x)$.

Eks:

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3 + C\right]_0^1$$

$$= \left(\tfrac{1}{3}\cdot 1^3 + \cancel{C}\right) - \left(\tfrac{1}{3}\cdot 0^3 + \cancel{C}\right)$$

$$= \tfrac{1}{3} - 0 = \underline{\underline{\tfrac{1}{3}}}$$

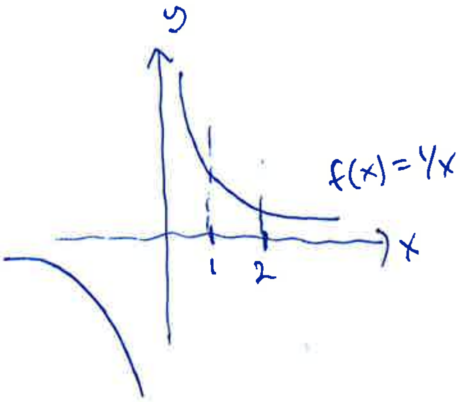
Merk: Det bestemte integralet $\int_a^b f(x)\,dx$ gir et tall, og tallet er uavhengig av hvilken antiderivert vi bruker.

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \underline{\underline{\tfrac{1}{3}}}$$

Eks:

$$\int_1^2 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1$$

$$= \ln 2 \simeq \underline{0{,}693}$$



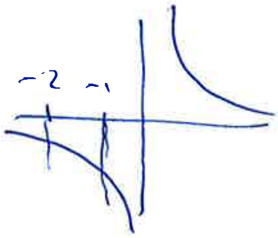
Fakta: Hvis $f(x)$ er kontinuerlig på intervallet $[a,b]$ så er

$$\int_a^b f(x)\,dx$$

definert.

Eks:

$$\int_{-2}^{-1} \frac{1}{x}\,dx$$



$$= \Big[\ln|x|\Big]_{-2}^{-1}$$

$$= \ln|-1| - \ln|-2|$$

$$= \ln 1 - \ln 2 = \underline{-\ln 2} \simeq \underline{-0{,}693}$$

Eks:

$$\int_0^1 x\ln(x^2+1)\,dx =$$

$$\boxed{\begin{array}{c} u = x^2+1 \\ du = 2x\,dx \end{array}} \qquad \begin{array}{l} x=0:\ u=1 \\ x=1:\ u=2 \end{array}$$

Alt I.

$$\int x\cdot\ln(x^2+1)\,dx = \int \cancel{x}\cdot\ln(u)\cdot\frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\cdot\int \ln(u)\,du = \frac{1}{2}\big(u\cdot\ln(u) - u\big) + C$$

$\uparrow$
delvis
int.

$$= \frac{1}{2}u\ln u - \frac{1}{2}u + C = \underline{\frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}(x^2+1) + C}$$

$$\int_0^1 x\ln(x^2+1)\,dx = \Big[\frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}(x^2+1)\Big]_0^1$$

$$= \big(1\cdot\ln(2) - 1\big) - \big(0 - \tfrac{1}{2}\big)$$

$$= \ln(2) - 1 + \frac{1}{2} = \underline{\ln(2) - \frac{1}{2}}$$

$$\simeq \underline{0{,}193}$$

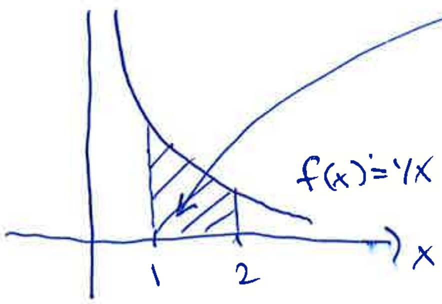
Alt 2:

$$= \int_1^2 \frac{1}{2}\cdot\ln|u|\,du = \Big[\frac{1}{2}(u\cdot\ln u - u)\Big]_1^2 = \frac{1}{2}(2\ln 2 - 2) - \frac{1}{2}(1\cdot\ln 1 - 1)$$

$$= \ln 2 - 1 + \frac{1}{2} = \underline{\ln 2 - 1/2} \simeq 0{,}193$$

Tolkning av det bestemte integralet

Eks: $$\int_1^2 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2 \simeq 0{,}693$$

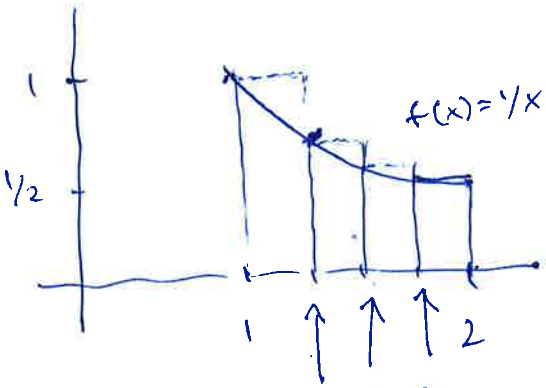


Arealet av området under grafen til $f(x) = 1/x$ på intervallet $[1, 2]$.

R: Området begrenset av $x=1$, $x=2$, grafen til f, x-aksen

A = areal av R
$= \ln(2) \simeq 0{,}693$

Hva betyr arealet under en graf?



$$\frac{5}{4} = 1{,}25 \quad \frac{6}{4} = 1{,}5 \quad \frac{7}{4} = 1{,}75$$

$n = 4$.

$$\text{Areal} \simeq \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot\frac{1}{5/4} + \frac{1}{4}\cdot\frac{1}{6/4} + \frac{1}{4}\cdot\frac{1}{7/4}$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \simeq 0{,}760$$

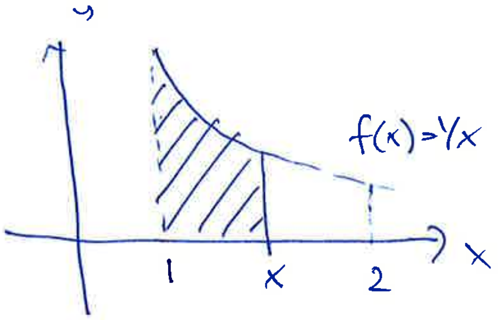
Definisjon: Riemannsum

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$$

bredden til den delintervall.

høyden målt i starten av hvert delint.

Hvorfor er arealet lik $\int_1^2 \frac{1}{x}\,dx$?



Arealfunksjonen:

$A(x)$ = arealet av området begrenset av grafen til $f(x)$ og x-aksen på $[1, x]$.

$$A(1) = 0$$
$$A(2) = A$$

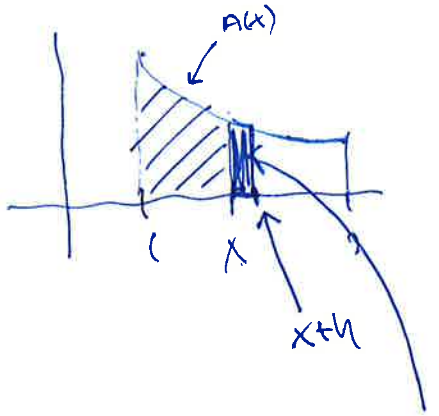
Fundamentalteoremet for analysen:

$$\boxed{A'(x) = f(x)}$$

① **Hvorfor er $A'(x) = f(x)$:**

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$A(x+h) - A(x)$:



$A(x+h) - A(x)$ = arealet av stripen på $[x, x+h]$

$$\simeq h \cdot f(x)$$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$
$$= \frac{h f(x)}{h} = f(x)$$

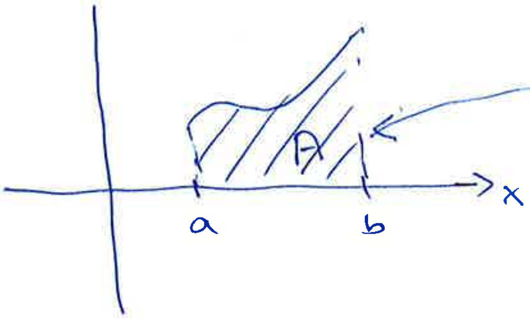
② **Konsekvenser av $A'(x) = f(x)$:**

$$\int_a^b f(x)\,dx = \Big[A(x)\Big]_a^b = A(b) - A(a)$$
$$= A - 0 = \underline{A}$$

Arealberegning:

Hvis f er en kontinuerlig funksjon på $[a,b]$ og $f(x) \geq 0$ på $[a,b]$, da er arealet A under grafen til f på $[a,b]$ gitt ved

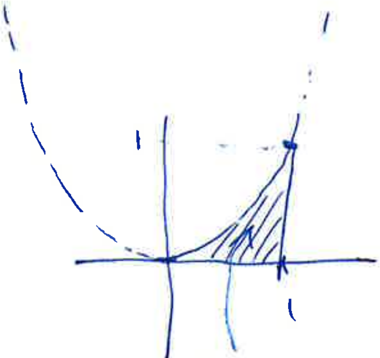
$$A = \int_a^b f(x)\,dx$$



$$A = \int_a^b f(x)\,dx$$

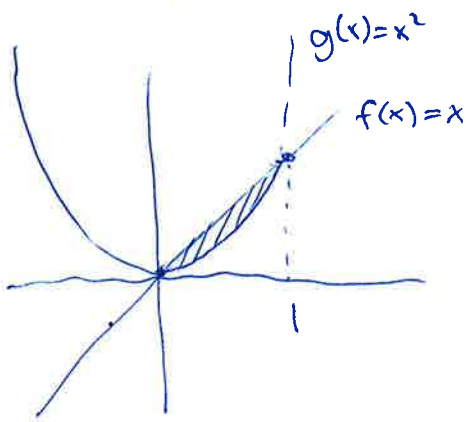
Eks:

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}\cdot 1 - \tfrac{1}{3}\cdot 0 = \underline{\tfrac{1}{3}}$$



Areal $= 1/3$

Eks: Finn arealet A av området begrenset av $f(x)=x$ og $g(x)=x^2$



$$x = x^2$$
$$0 = x^2 - x$$
$$0 = x\cdot(x-1)$$
$$\underline{x=0}, \quad \underline{x=1}$$

$$A = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 - \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3}$$

$$= \frac{3}{6} - \frac{2}{6} = \underline{\frac{1}{6}} \simeq 0{,}167$$