FORELESNING 24

MET 1180 BI

MATEMATIKK

EIVIND ERIKSEN, MAR 09, 2016

Plan:

1. Inverse matriser
2. Eksamensoppgave
   MET11803 12/2015, Oppg. 3

Pensum:

[S] 10.10 - 10.11
[E] 6.6

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(1) Inverse matriser

A

Kvadratisk matrise ($n \times n$)

Defn: Hvis det finnes en matrise B slik at

$$A \cdot B = I \quad \text{og} \quad B \cdot A = I$$

da kalles B en invers matrise til A, og skrives $A^{-1} = B$.
Hvis A^{-1} finnes, så er den entydig.

Tilfellet $n=2$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \qquad A^{-1} = \begin{cases} \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, & |A| \neq 0 \\ \text{finnes ikke}, & |A| = 0 \end{cases}$$

Eks:

$$A = \begin{pmatrix} 7 & 3 \\ -1 & 4 \end{pmatrix} \qquad A^{-1} = ?$$

$$|A| = 7 \cdot 4 - 3 \cdot (-1)$$
$$= 28 + 3 = \underline{31} \neq 0$$

$$A^{-1} = \frac{1}{31} \cdot \begin{pmatrix} 4 & -3 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 4/31 & -3/31 \\ 1/31 & 7/31 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)}$$

adj(A): den adjungerte matrisen til A

Sjekk:

$$A^{-1} \cdot A = \underbrace{\frac{1}{31} \cdot \begin{pmatrix} 4 & -3 \\ 1 & 7 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 7 & 3 \\ -1 & 4 \end{pmatrix}}_{A}$$

$$= \frac{1}{31} \cdot \begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

For reelle tall:

a tall: $a^{-1} = 1/a$ er det tallet $\underset{a^{-1}}{\underset{\shortparallel}{b}}$ slik at $ab = 1$
(hvis $a \neq 0$)

a^{-1} finnes ikke hvis $a = 0$

Eks: Lineære systemer

$$7x + 3y = 4$$
$$-x + 4y = 5$$

Matriseform: $A \cdot \underline{x} = \underline{b}$

$$A = \begin{pmatrix} 7 & 3 \\ -1 & 4 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Løsning: $A\underline{x} = \underline{b} \quad | A^{-1}\cdot$

$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b}$$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -1 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$= \frac{1}{31} \cdot \begin{pmatrix} 4 & -3 \\ 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{31} \cdot \begin{pmatrix} 1 \\ 39 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/31 \\ 39/31 \end{pmatrix}}}$$

Husk: $A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$

fungerer __kun__ om A er kvadratisk og $|A| \neq 0$

BI

Tilfellet $n > 2$:

A kvadratisk matrise $(n \times n)$

Resultat:

Hvis $|A| \neq 0$, er $A^{-1} = \frac{1}{|A|} \cdot adj(A)$

Hvis $|A| = 0$, finnes ikke A^{-1}

Eks:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$|A| = 1 \cdot 6 - 1 \cdot 6 + 1 \cdot 2 = \underline{2} \neq 0$$

$$\Rightarrow A^{-1} = \underline{\frac{1}{2} \cdot adj(A)}$$

Adjungert matrise:

Kofaktormatrisen:

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

I Eks $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$:

$$\left.\begin{matrix} C_{11} = 6 & C_{12} = -5 & C_{13} = 1 \\ C_{21} = -6 & C_{22} = 8 & C_{23} = -2 \\ C_{31} = 2 & C_{32} = -3 & C_{33} = 1 \end{matrix}\right\} \quad C = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Adjungert matrise: $adj(A) = C^T$

$$adj(A) = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Alternativ metode for å regne ut A^{-1}

$$(A \mid I) \to \cdots \to (I \mid A^{-1})$$

redusert trappeform

Eks: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -1 \\ -1 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -2 \end{array} \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \begin{array}{l} \\ \\ :2 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{array}{l} -1 \\ -3 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{array}{l} \\ -1 \\ \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Hvis første del av reduserte trappeform ikke er I, så fins ikke A^{-1}!

Eksamen MET11803 12/2015, Opps. 3

$$\begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ a \\ 2 \end{pmatrix}$$

a) $A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}$

$$|A| = \begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = a \cdot (a^2-4) - 2(2a-4) + 2(4-2a)$$

Alt 1

$$= a^3 - 4a - 4a + 8 + 8 - 4a$$

$$= \underline{\underline{a^3 - 12a + 16}}$$

Alt 2:

$$a \cdot (a^2-4) - 2(2a-4) + 2(4-2a)$$

$$= a \cdot (a^2-4) - 8a + 16$$

$$= a \cdot (a-2)(a+2) - 8(a-2)$$

$$= (a-2) \cdot [a \cdot (a+2) - 8]$$

$$= \underline{(a-2) \cdot (a^2 + 2a - 8)}$$

$$= \underline{(a-2)(a+4)(a-2)}$$

$$= (a-2)^2 \cdot (a+4)$$

$$a^2 + 2a - 8 = 0$$

$$a = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= \frac{-2 \pm 6}{2}$$

$$= -4, 2$$

b) $\underline{a=0}$: $$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$|A| = a^3 - 12a + 16 \underset{\substack{\uparrow \\ a=0}}{=} 16 \neq 0$$

$$A^{-1} = \frac{1}{16} \cdot \operatorname{adj}(A) = \underline{\underline{\frac{1}{16} \cdot \begin{pmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 4 & 4 & -4 \end{pmatrix}}}$$

$$\operatorname{adj}(A) = \underbrace{\begin{pmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 4 & 4 & -4 \end{pmatrix}}_{C}^{T} = \begin{pmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 4 & 4 & -4 \end{pmatrix}$$

c) Uendelig mange løsninger: $A\underline{x} = \underline{b}$

$$\begin{aligned} ax + 2y + 2z &= 2 \\ 2x + ay + 2z &= a \\ 2x + 2y + az &= 2 \end{aligned}$$

Metode 1: $\det A \neq 0 \longrightarrow$ én løsn.

$\det A = 0 \longrightarrow$ uendelig mange løsn. eller ingen løsn.

$$\det(A) = a^3 - 12a + 16 = 0$$

$$= (a-2)^2 \cdot (a+4) = 0$$

$$\Updownarrow$$

$\underline{a=2}$ eller $\underline{a=-4}$

$\underline{a=2:}$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{array}\right) \begin{array}{c} \\ -1 \\ -1 \end{array} \to \left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn.
y, z fri

$$2x + 2y + 2z = 2 \quad | : 2$$

$$x + y + z = 1$$

$$x = \underline{1 - y - z}, \quad y \text{ fri}, \; z \text{ fri}$$

$$x = 1 - y - z$$
$$y = \text{fri}$$
$$z = \text{fri}$$

$\underline{a=-4}$:

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 2 \\ 2 & -4 & 2 & -4 \\ 2 & 2 & -4 & 2 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -4 & 2 \\ 2 & -4 & 2 & -4 \\ -4 & 2 & 2 & 2 \end{array}\right) \begin{array}{l} \cdot(-1) \\ \\ \cdot 2 \end{array} \longrightarrow$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -4 & 2 \\ 0 & -6 & 6 & -6 \\ 0 & 6 & -6 & 6 \end{array}\right) \begin{array}{l} \\ \\ \cdot 1 \end{array} \longrightarrow$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -4 & 2 \\ 0 & -6 & 6 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Uendelig mange løsn. (z fri)

$$\underline{2x} + 2y - 4z = 2 \qquad x = 1 - y + 2z = \underline{z}$$

$$-6\underline{y} + 6z = -6 \implies y = \underline{z+1}$$

$x = z$
$y = z+1$
$z =$ fri

Konklusjon:

Uend. mange løsn: $\underline{a = 2, -4}$

(Én løsn : $a \neq 2, -4$)

d) $\underline{a \neq 2, -4:}$ $|A| \neq 0$

En løsning.

$\underline{x} = A^{-1} \cdot \underline{b}$ eller

Cramers regel

$$x = \frac{|A_1(\underline{b})|}{|A|}$$

$$y = \frac{|A_2(\underline{b})|}{|A|}$$

$$z = \frac{|A_3(\underline{b})|}{|A|}$$

eller Gauss

$$|A| = \begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = a^3 - 12a + 16 = (a-2)^2(a+4)$$

$$|A_1(\underline{b})| = \begin{vmatrix} 2 & 2 & 2 \\ a & a & 2 \\ 2 & 2 & a \end{vmatrix} = 2 \cdot 0 - 2 \cdot 0 + a \cdot 0 = 0$$

$\uparrow$ $\underline{b}$

$$\Downarrow$$

$$x = \frac{|A_1(\underline{b})|}{|A|} = \frac{0}{(a-2)^2(a+4)} = \underline{\underline{0}}$$

$$|A_2(\underline{b})| = \begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = |A| = (a-2)^2(a+4)$$

$\uparrow$ $\underline{b}$

$$\Downarrow$$

$$y = \frac{(a-2)^2(a+4)}{(a-2)^2(a+4)} = \underline{\underline{1}}$$

$$|A_3(b)| = \begin{vmatrix} a & 2 & 2 \\ 2 & a & a \\ 2 & 2 & 2 \end{vmatrix} = 0 \Rightarrow z = \frac{|A_3(b)|}{|A|} = \frac{0}{|A|} = \underline{0}$$

$\underline{\text{Konl:}}$ $(x, y, z) = \underline{\underline{(0, 1, 0)}}$ for $a \neq 2, -4$