Plan

1 Repetisjon og oppgavegjennomgang
2 Anvendelser av integrasjon

① Repetisjon, Oppgaveark 28

Repetisjon: – Integrasjon av rasjonale uttrykk

* polynomdivisjon (grad til teller "for høy")
* Substitusjon
* delbrøksoppspalting

$$\boxed{\text{Eks: } \frac{x}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}}$$

– bestemte integral

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ der } F'(x) = f(x)$$

Oppgaveark 28:

2d) $$\int \frac{x^2-2x+1}{1-x^2}\,dx = \int -1 + \frac{-2x+2}{1-x^2}\,dx$$

$$\boxed{\begin{array}{l} x^2-2x+1 : -x^2+1 = \underline{-1} \\ \underline{x^2 \quad\quad -1} \\ -2x+2 \end{array}}$$

$$= -x + \int \frac{2\cancel{(1-x)}}{\cancel{(1-x)}(1+x)}\,dx = -x + \int \frac{2}{1+x}\,dx$$

$$= -x + 2\ln|1+x| + C$$

2f) $$\int \frac{2x}{(1-x)^2}\,dx = \int \frac{-2}{1-x} + \frac{2}{(1-x)^2}\,dx$$

$$\boxed{\begin{array}{l} u = 1-x \\ du = -1\cdot dx \end{array}}$$

$$\boxed{\begin{array}{l} \frac{2x}{(1-x)^2} = \frac{\overset{-2}{\overset{\shortparallel}{A}}}{1-x} + \frac{\overset{2}{\overset{\shortparallel}{B}}}{(1-x)^2} \\ 2x = A\cdot(1-x) + B \\ \quad = \underset{\overset{\shortparallel}{0}}{(A+B)} + \underset{\overset{\shortparallel}{2}}{(-A)}x \\ B = 2 \quad A = -2 \end{array}}$$

$$= \frac{-2}{-1}\cdot \ln|1-x| + \int \frac{2}{u^2}\cdot\frac{1}{-1}\,du$$

$$\boxed{\int u^{-2}\,du = \frac{u^{-1}}{-1} + C}$$

$$= 2\ln|1-x| - 2\left(\frac{1}{-1}\right)\cdot\frac{1}{u} + C$$

$$= 2\ln|1-x| + \frac{2}{1-x} + C$$

3j) $\int_{-1}^{1} e^x + e^{-x}\,dx = \left[e^x - e^{-x}\right]_{-1}^{1} = (e^1 - e^{-1}) - (e^{-1} - e^1)$

$= \underline{\underline{2e - \frac{2}{e}}} = 2(e - 1/e)$

4a) $\int_0^1 x e^x\,dx = \left[x e^x\right]_0^1 - \int_0^1 e^x\,dx = \left[x e^x - e^x\right]_0^1$

$\boxed{\begin{matrix} u = e^x & v = x \\ u' = e^x & v' = 1 \end{matrix}}$

$= (\cancel{1 \cdot e^1 - e^1}) - (\cancel{0 \cdot e^0} - e^0) = \underline{\underline{1}}$

4b) $\int_0^1 x \cdot \ln(x^2+1)\,dx = \int_1^2 \cancel{x} \cdot \ln(u) \cdot \frac{1}{2\cancel{x}}\,du = \int_1^2 \frac{1}{2} \ln(u)\,du$

$\boxed{\begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix}}$ $x=0:\ u=1$, $x=1:\ u=2$

$\boxed{\begin{aligned} &\int \ln(x)\,dx \\ &= x \cdot \ln x - \int 1\,dx \\ &= x\ln x - x + C \end{aligned}}$

$= \left[\frac{1}{2}(u \cdot \ln u - u)\right]_1^2$

$= \frac{1}{2}(2\ln 2 - 2) - \frac{1}{2}(\cancel{1 \cdot \ln(1)} - 1)$

$= \ln 2 - 1 + \frac{1}{2} = \underline{\underline{\ln 2 - 1/2}} = \frac{1}{2}(2\ln 2 - 1)$

4d) $\int_0^1 \frac{1}{x^2+4x+4}\,dx = \int_0^1 \frac{1}{(x+2)^2}\,dx = \int_2^3 u^{-2}\,du$

$\boxed{\begin{matrix} u = x+2 \\ du = dx \end{matrix}}$

$= \left[-\frac{1}{u}\right]_2^3 = (-1/3) - (-1/2) = \frac{1}{2} - \frac{1}{3} = \underline{\underline{1/6}}$

6b) $\int x^3 \sqrt{x^2+4}\, dx = \int x^3 \cdot \sqrt{u}\, \frac{1}{2x}\, du$

$$\boxed{\begin{array}{l} u = x^2+4 \\ du = 2x\, dx \end{array}} \to x^2 = u-4$$

$$= \int \frac{1}{2} x^2 u^{1/2}\, du = \int \frac{1}{2}(u-4)\, u^{1/2}\, du = \frac{1}{2}\int u^{3/2} - 4u^{1/2}\, du$$

$$= \frac{1}{2}\left(\frac{2}{5} \cdot u^{5/2} - 4 \cdot \frac{2}{3} \cdot u^{3/2} \right) + C$$

$$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

7c) $\int \frac{\sqrt{x}+1}{1-\sqrt{x}}\, dx = \int \frac{\sqrt{x}+1}{u} \cdot (-2\sqrt{x})\, du$

$$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\, dx \end{array}} \to \sqrt{x} = 1-u$$

$$x = (\sqrt{x})^2 = (1-u)^2$$

$$\Downarrow$$

$$dx = -2\sqrt{x}\, du$$

$$= \int \frac{(\sqrt{x}+1)(-2\sqrt{x})}{u}\, du = \int \frac{-2x - 2\sqrt{x}}{u}\, du$$

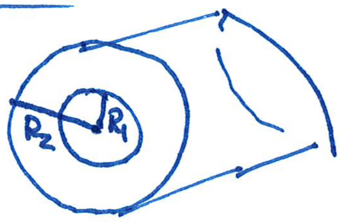
$$= \int \frac{-2(1-u)^2 - 2(1-u)}{u}\, du = \int \frac{-2(1-2u+u^2) - 2(1-u)}{u}\, du$$

$$= \int \frac{-2u^2 + 6u - 4}{u}\, du = \int -2u + 6 - \frac{4}{u}\, du$$

$$= -u^2 + 6u - 4\ln|u| + C = -(1-\sqrt{x})^2 + 6(1-\sqrt{x}) - 4\ln|1-\sqrt{x}| + C$$

2) Anvendelse av integrasjon

Eks:



$R_1 = 2\,\text{cm}$
$R_2 = 7\,\text{cm}$
$\tau = 0.05\,\text{cm}$ (tykkelsen på papiret)

Diskret:

(1) $2\pi \cdot R_1 + 2\pi \cdot (R_1+\tau) + 2\pi(R_1+2\tau) + \dots + 2\pi R_2$

$$= \frac{\cancel{2}\pi R_1 + \cancel{2}\pi R_2}{\cancel{2}} \cdot \frac{R_2 - R_1}{\tau}$$

$$= \pi \cdot (R_1+R_2) \cdot \frac{R_2-R_1}{\tau} = \frac{\pi}{\tau}(R_2^2 - R_1^2)$$

Kontinuerlig:

(2) **Funksjon** $\ell(r)$ = lengden som er igjen på rullen når radius er r, $R_1 \le r \le R_2$

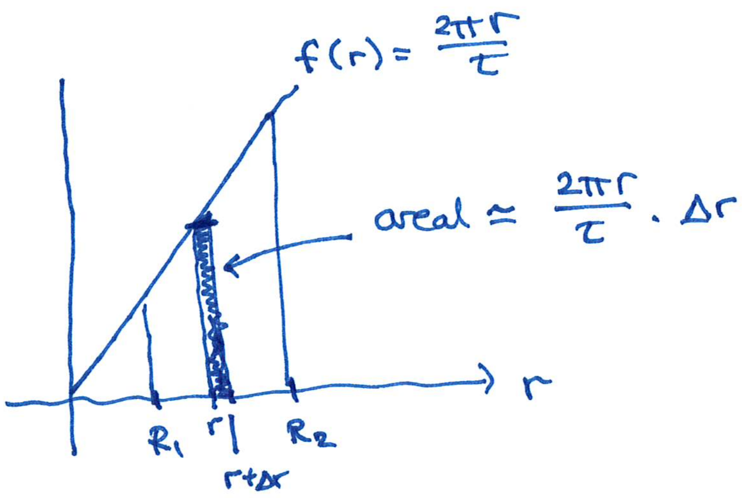
Formel ved integrasjon:

$$\ell = \frac{\pi}{\tau}(R_2^2 - R_1^2)$$
$$= \frac{\pi}{0.05}(7^2 - 2^2)$$
$$= \frac{45\pi}{0.05} = 900\pi$$
$$\simeq 28.3\,\text{m}$$

Ser på et lite intervall $[r, r+\Delta r]$

$$\Delta \ell = \ell(r+\Delta r) - \ell(r) = 2\pi r \cdot \frac{\Delta r}{\tau} = \frac{2\pi r}{\tau} \cdot \Delta r$$

$$\frac{\Delta \ell}{\Delta r} = \frac{2\pi r}{\tau} \longleftarrow \simeq \ell'(r)$$



Hovedpoeng:

lengden av rullen

= summen av $\frac{2\pi r}{\tau} \cdot \Delta r$ for mange små intervall $[r, r+\Delta r]$ fra R_1 til R_2

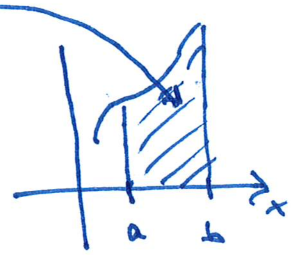
$$= \int_{R_1}^{R_2} \frac{2\pi r}{\tau}\,dr = \frac{\cancel{2}\pi}{\tau}\left[\tfrac{1}{\cancel{2}} r^2\right]_{R_1}^{R_2}$$

$$= \frac{\pi}{\tau}(R_2^2 - R_1^2)$$

Teorem:

Hvis $f(x)$ er en kontinuerlig funksjon på intervallet $[a,b]$ og $f(x) \geq 0$ for $a \leq x \leq b$, så er

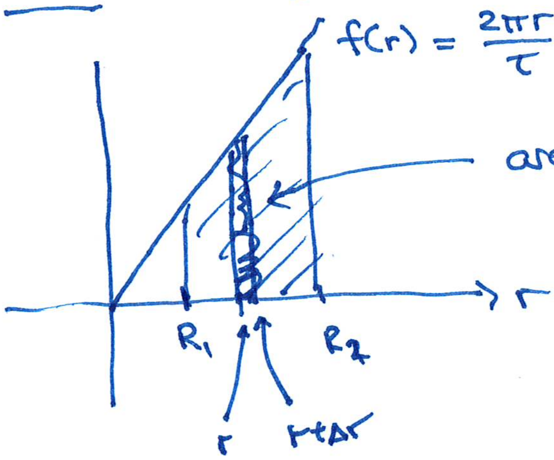
$$\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \text{arealet under grafen} \\ \text{til } y = f(x) \text{ i} \\ \text{intervallet } [a,b] \end{array} \right\}$$



Eks: Lengden av en toalettrull

$$f(r) = \frac{2\pi r}{\tau}$$

areal = lengden av toalettrullen



$$\Delta l = \frac{2\pi r}{\tau} \cdot \Delta r \rightarrow l'(r) = \frac{\Delta l}{\Delta r} = \frac{2\pi r}{\tau}$$