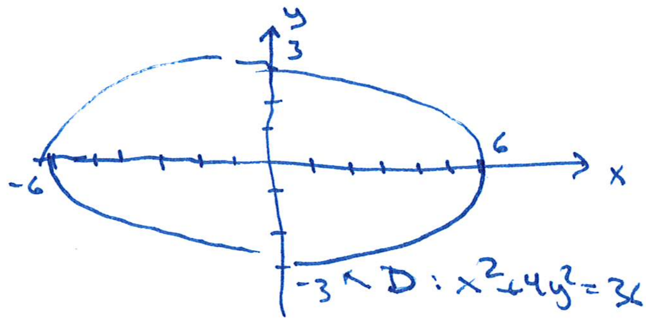
Plan

1. Lagranges multiplikatormetode
2. Nødvendige betingelser for maksimum

① Lagranges multoplikatormetode

Ex: max $f(x,y) = x + 2y$

når $x^2 + 4y^2 = 36$



Merk: D er lukket (=) og begrenset

$(-6 \leq x \leq 6,\ -3 \leq y \leq 3$ for alle (x,y) i D$)$

ekstremverdi-setningen $\Downarrow$

det fins et max (for f på D)

Merk: D består kun av randpkt.

$$x^2 + 4y^2 = 36 \quad | : 36$$

$$\frac{x^2}{36} + \frac{4y^2}{36} = 1$$

$$\boxed{\frac{x^2}{36} + \frac{y^2}{9} = 1}$$

ellipse, senter i (0,0),

$a = 6 \quad b = 3 \quad \leftarrow$ halvakser

Lagranges multiplikatormetode: max $f(x,y) = x + 2y$

når $\underbrace{x^2+4y^2}_{g(x,y)} = \underbrace{36}_{a}$

$$L(x,y;\lambda) = f(x,y) - \lambda \cdot (g(x,y) - a)$$

$$= x + 2y - \lambda(x^2 + 4y^2 - 36)$$

* Når bibetingelsen er en likning, så kan den alltid skrives $g(x,y) = a$.

$$\left.\begin{aligned} L'_x &= f'_x - \lambda \cdot g'_x = 1 - \lambda(2x) = 0 \\ L'_y &= f'_y - \lambda \cdot g'_y = 2 - \lambda(8y) = 0 \end{aligned}\right\} \text{FOC (førsteordens-betingelser)}$$

$$\left.\begin{array}{c} L'_\lambda = -(g(x,y)-a) = -(x^2+4y^2-36) = 0 \\ \Updownarrow \\ g(x,y)-a = x^2+4y^2-36 = 0 \\ \Updownarrow \\ g(x,y)=a \text{ eller } x^2+4y^2=36 \end{array}\right\} \text{C (bibetingelsen)}$$

<u>Lagrange-betingelsene:</u> FOC+C $\left\{\begin{array}{l} L'_x = 0 \\ L'_y = 0 \\ g(x,y)=a \end{array}\right\}$ 3×3 likn.-system

<u>Prinsipp:</u> Løsninger $(x,y;\lambda)$ av FOC+C $\simeq$ kandidatpkt

<u>I eks:</u>
$$\boxed{\begin{array}{r} 1-\lambda\cdot 2x = 0 \\ 2-\lambda\cdot 8y = 0 \\ x^2+4y^2 = 36 \end{array}} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{FOC} \\ \} \text{ C} \end{array} \quad \begin{array}{l} L'_x=0 \\ L'_y=0 \\ g(x,y)=a \end{array}$$

<u>Alt I:</u>

(1) $1 = 2\lambda x \quad | : 2\lambda$

$\underline{x = \frac{1}{2\lambda}} \quad (\lambda \neq 0)$

(2) $2 = 8\lambda \cdot y \quad | : 8\lambda$

$y = \frac{2}{8\lambda} = \underline{\frac{1}{4\lambda}} \quad (\lambda \neq 0)$

(3) $x^2+4y^2 = 36$

$\left(\frac{1}{2\lambda}\right)^2 + 4\cdot\left(\frac{1}{4\lambda}\right)^2 = 36$

$\frac{1}{4\lambda^2} + \frac{\cancel{4}\cdot 1}{4\cancel{16}\lambda^2} = 36$

$\frac{2}{4\lambda^2} = 36 \quad | \cdot 4\lambda^2$

$\rightarrow 2 = 36\cdot 4\lambda^2 \quad | : 4\cdot 36$

$\lambda^2 = \frac{\cancel{2}}{36\cdot\cancel{4}\,2} = \frac{1}{72}$

$\lambda = \pm\sqrt{\frac{1}{72}} = \pm\frac{1}{\sqrt{72}}$

<u>Kandidatpkt:</u>

$\lambda = 1/\sqrt{72}: \quad x = \frac{1\cdot\sqrt{72}}{2\cdot 1/\sqrt{72}\cdot\sqrt{72}}$

$= \frac{\sqrt{72}}{2} = \frac{\sqrt{36}\cdot\sqrt{2}}{2}$

$= \underline{3\sqrt{2}}$

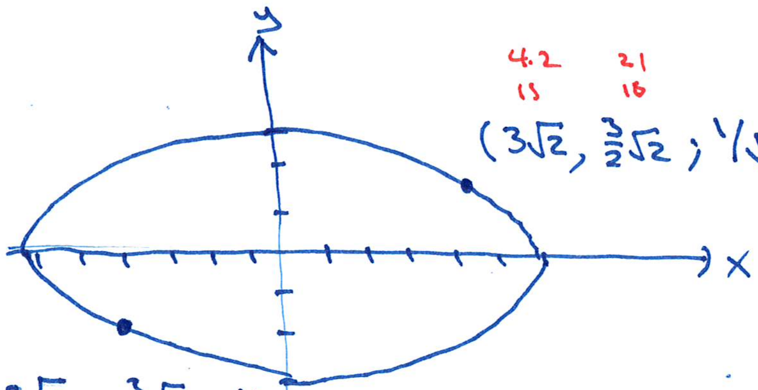
$y = \frac{1}{4\lambda} = \frac{1}{2}\left(\frac{1}{2\lambda}\right) = \frac{1}{2}\cdot 3\sqrt{2}$

$= \underline{\frac{3}{2}\sqrt{2}}$

$$\Rightarrow (x,y;\lambda) = \underline{\left(3\sqrt{2}, \tfrac{3}{2}\sqrt{2}; 1/\sqrt{72}\right)},$$

$$\underline{\left(-3\sqrt{2}, -\tfrac{3}{2}\sqrt{2}; -1/\sqrt{72}\right)}$$

$\lambda = -\frac{1}{\sqrt{72}}$ istedet for $\lambda = 1/\sqrt{72} \Rightarrow$ x og y bytter fortegn



$\left(3\sqrt{2}, \tfrac{3}{2}\sqrt{2}; 1/\sqrt{72}\right)$ (4.2, 2.1 ; 1.5, 1.6) $\quad f = 3\sqrt{2} + \cancel{2}\cdot\frac{3}{\cancel{2}}\sqrt{2} = \underline{6\sqrt{2}}$ (8.4, 1.3)

$\left(-3\sqrt{2}, -\tfrac{3}{2}\sqrt{2}; -1/\sqrt{72}\right)$ (1.5, 1.5 ; -4.2, -2.1) $\quad f = -3\sqrt{2} + \cancel{2}\cdot\left(-\frac{3}{\cancel{2}}\sqrt{2}\right) = \underline{-6\sqrt{2}}$ (-8.4, 1.5)

Konklusjon: $f_{max} = \underline{\underline{6\sqrt{2}}}$ max-pkt: $(x,y) = \left(3\sqrt{2}, \tfrac{3}{2}\sqrt{2}\right)$

Lagrange-multiplikator: $\lambda = 1/\sqrt{72}$

Alt 2:

$$\boxed{\begin{aligned} 1 - \lambda\cdot 2x &= 0 \\ 2 - \lambda\cdot 8y &= 0 \\ x^2 + 4y^2 &= 36 \end{aligned}}$$

(1) $1 = 2\lambda\cdot x \Rightarrow \underline{\lambda = \frac{1}{2x}}$ $(x \neq 0)$

(2) $2 = 8\lambda y \Rightarrow \lambda = \underline{\frac{2}{8y} = \frac{1}{4y}}$ $(y \neq 0)$

$\Rightarrow \frac{1}{2x} = \frac{1}{4y} \quad | \cdot 4xy$

$\frac{2\cancel{4xy}}{\cancel{2x}} = \frac{\cancel{4xy}}{\cancel{4y}}$ $\qquad 2y = x$, $\underline{x = 2y}$

(3) $x^2 + 4y^2 = 36$

$(2y)^2 + 4y^2 = 36$

$4y^2 + 4y^2 = 36$

$8y^2 = 36$

$y^2 = \frac{36}{8} = \frac{9}{2}$

$y = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}}$

$= \pm\frac{3}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{3}{2}\sqrt{2}$

Kandidatpkt:

1) $y = \frac{3}{2}\sqrt{2}$

$x = 2y = 3\sqrt{2}$

$\lambda = \frac{1}{2\cdot 3\sqrt{2}} = \frac{1}{6\sqrt{2}} = \frac{1}{\sqrt{72}}$

$(x,y;\lambda) = \left(3\sqrt{2}, \tfrac{3}{2}\sqrt{2}; \tfrac{1}{\sqrt{72}}\right)$

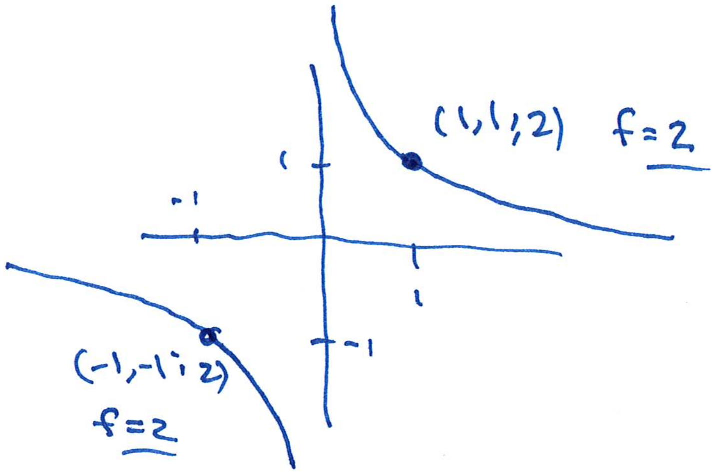
2) $y = -\frac{3}{2}\sqrt{2}$

$x = -3\sqrt{2}$

$\lambda = -1/\sqrt{72}$

$\left(-3\sqrt{2}, -\tfrac{3}{2}\sqrt{2}; -1/\sqrt{72}\right)$

Eks: min $f(x,y) = x^2+y^2$ når $xy=1$



D: $xy=1$

$y=1/x$

D: lukket (=), men ikke begrenset

Alt I:

$\mathcal{L} = x^2+y^2 - \lambda\cdot(xy-1)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 2x - \lambda y = 0 \\ \mathcal{L}'_y &= 2y - \lambda x = 0 \\ & xy = 1 \end{aligned}}$$

(1) $2x = \lambda y$

$x = \lambda y/2$

(2) $2y = \lambda x = \lambda\cdot(\lambda y/2) = \frac{\lambda^2}{2}\cdot y$

$\cancel{y = \lambda x/2}$

$2y - \frac{\lambda^2}{2}y = 0 \quad |\cdot 2$

$4y - \lambda^2 y = 0$

$y(4-\lambda^2) = 0$

$y=0$ eller $4-\lambda^2=0$

$y=0$	$\lambda^2=4$	
$x=0$	$\lambda=2$	$\lambda=-2$
$0\cdot 0=1$	$x=y$	$x=-y$
umulig	$xy=1$	$(-y)\cdot y=1$
	$x^2=1$	$-y^2=1$
	$x=\pm 1$	$y^2=-1$
		umulig
	$(1,1;2)$,	
	$(-1,-1;2)$	

Alt. 2: $xy=1 \Rightarrow y=1/x$

$f(x,y)=x^2+y^2 \quad f(x,1/x) = x^2+(1/x)^2$

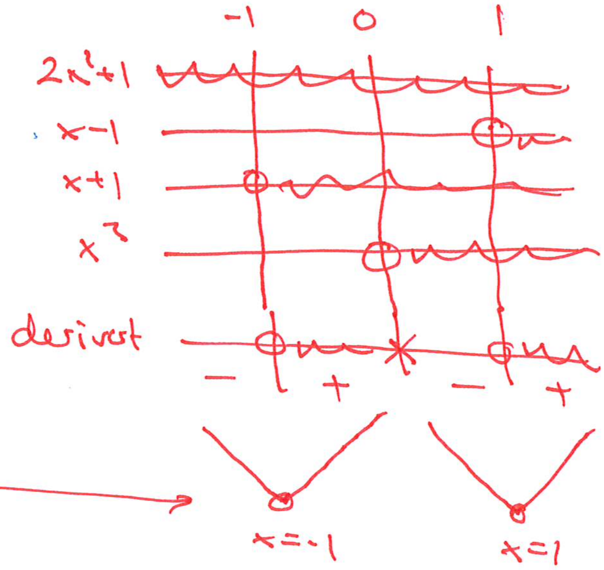
$\Downarrow$

min $f(x,1/x) = x^2 + 1/x^2$

$$\left(x^2+1/x^2\right)' = \left(x^2+x^{-2}\right)' = 2x + (-2)x^{-3} = 2x - \frac{2}{x^3}$$

$$= \frac{2x\cdot x^3-2}{x^3} = \frac{2x^4-2}{x^3} = \frac{2(x^4-1)}{x^3} = \frac{2(x^2-1)(x^2+1)}{x^3}$$

$$= \frac{2(x^2+1)(x-1)(x+1)}{x^3}$$



$\boxed{y=1/x}$

$\downarrow$

$$\left.\begin{array}{l} x=1: \quad f(1, 1/1) = f(1,1) = 2 \\ x=-1: \quad f(-1, 1/(-1)) = f(-1,-1) = 2 \end{array}\right\} \longrightarrow$$

$x=1$, $x=-1$ er <u>minimumspkt</u>

$\Rightarrow f_{min} = \underline{\underline{2}}$ i $(x,y) = \underline{\underline{(1,1), (-1,-1)}}$ $\leftarrow$ Kandidatpkt i Lagrange