Løsning: MET 11806 03/2023

1. a) $\int_0^7 x^2\sqrt{x}\,dx = \int_0^7 x^{5/2}\,dx = \left[\frac{2}{7}x^{7/2}\right]_0^7 = \frac{2}{7}\left(7^3\cdot\sqrt{7}-0\right) = 2\cdot 7^2\sqrt{7} = \underline{98\sqrt{7}}$

b) $\int_1^2 \ln(\sqrt{x})\,dx = \int_1^2 \frac{1}{2}\ln x\,dx = \frac{1}{2}\left[x\ln x - x\right]_1^2 = \frac{1}{2}(2\ln 2 - 2) - \frac{1}{2}(1\cdot\ln 1 - 1)$

$= \underline{\ln 2 - \frac{1}{2}}$

c) $\int_1^2 \frac{6}{x^2-9}\,dx = \int_1^2 \frac{1}{x-3} - \frac{1}{x+3}\,dx = \left[\ln|x-3| - \ln|x+3|\right]_1^2$

$= (\ln 1 - \ln 5) - (\ln 2 - \ln 4) = 0 - \ln 5 - \ln 2 + 2\ln 2$

$= \ln 2 - \ln 5 = \underline{\ln(2/5)}$

$\frac{6}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$6 = A(x+3) + B(x-3)$

$A = 1,\ B = -1$

d) $\int_0^1 \frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \int \frac{\sqrt{x}}{u}\,2\sqrt{x}\,du = \int \frac{2(\sqrt{x})^2}{u}\,du = \int_1^2 \frac{2(u-1)^2}{u}\,du$

$\boxed{u = \sqrt{x}+1,\quad du = \frac{1}{2\sqrt{x}}\,dx}$

$= \int_1^2 \frac{2(u^2-2u+1)}{u}\,du = \int_1^2 2u - 4 + 2/u\,du = \left[u^2 - 4u + 2\ln|u|\right]_1^2$

$= (4 - 8 + 2\ln 2) - (1 - 4 + 2\ln 1) = \underline{2\ln 2 - 1}$

c) $\int_{-1}^0 x\sqrt{-x}\,dx = \int x\sqrt{u}\,(-1)\,du = \int_1^0 -u\sqrt{u}\,(-1)\,du = \int_1^0 u^{3/2}\,du$

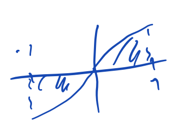
$\boxed{u = -x,\quad du = -dx}$

$= \left[\frac{2}{5}u^{5/2}\right]_1^0 = (0) - (2/5) = \underline{-2/5}$

f) $\int_{-1}^1 x\sqrt{|x|}\,dx = \int_{-1}^0 x\sqrt{-x}\,dx + \int_0^1 x\sqrt{x}\,dx = -2/5 + \left[\frac{2}{5}x^{5/2}\right]_0^1$

$= -5/2 + (2/5) - (0) = \underline{0}$

(Kan også se dette ved symmetri: $f(x) = x\sqrt{|x|}$ gir $f(-x) = -f(x)$ dvs:



$\int_{-1}^1 f(x)\,dx = -A + A = 0$)

g) $\displaystyle\int_1^{e^2} \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{\sqrt{u}}{x}\cdot x\,du = \int_0^2 \sqrt{u}\,du = \int_0^2 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^2$

$\boxed{\begin{array}{l} u = \ln x \\ du = 1/x\,dx \end{array}}$

$= \frac{2}{3}(2\sqrt{2}) - \frac{2}{3}(0) = \underline{\frac{4}{3}\sqrt{2}}$

2. a) P: $f(x) = a(x-2)^2 + 5$ sidan $\underline{x=2}$ symmetrilinje, $y=5$ topp-pkt.

$f(2\pm\sqrt{5}) = a(\pm\sqrt{5})^2 + 5 = 0$ ← nullpkt

$5a + 5 = 0$

$a = -1$

⇓

P: $f(x) = 5 - (x-2)^2 = \underline{1 + 4x - x^2}$

H: $(x-0)(y-0) = c$ sidan $x=0, y=0$ er asymptoter

$xy = c$

$y = c/x$

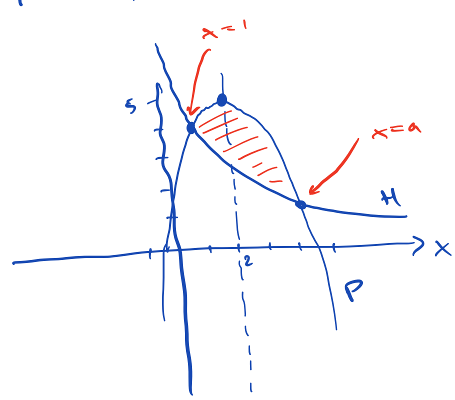
H: $g(x) = c/x$ ↙ Skjæringspkt i $x=1$:

$f(1) = g(1)$

$1 + 4\cdot 1 - 1^2 = c/1$ $c=4$

$g(x) = \underline{4/x}$

b) Areal $= \displaystyle\int_1^a f(x) - g(x)\,dx$



Finner a: $1 + 4x - x^2 = 4/x$ $|\cdot x$

$x + 4x^2 - x^3 = 4$

$x^3 - 4x^2 - x + 4 = 0$

$(x-1)(x^2 - 3x - 4) = 0$

$\underline{x=1}$ eller $x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$\underline{a=4}$ ⟶ $\underline{x=4}$, $\underline{x=-1}$

Areal: $\displaystyle\int_1^4 1 + 4x - x^2 - 4/x\,dx = \left[x + 2x^2 - \frac{1}{3}x^3 - 4\ln|x|\right]_1^4$

$= \left(4 + 2\cdot 16 - \frac{1}{3}\cdot 64 - 4\ln 4\right) - \left(1 + 2 - \frac{1}{3} - 4\ln 1\right)$

$= 4 + 32 - 3 - \frac{64}{3} + \frac{1}{3} - 4\ln 4$

$= 33 - \frac{63}{3} - 4\ln(2^2) = 33 - 21 - 8\ln 2 = \underline{12 - 8\ln 2}$

3. Total kontantstrøm:

$$\int_0^{25} f(t)\,dt = \int_0^{25} 100\, e^{\sqrt{t}}\,dt = \int 100\, e^{u} \cdot 2\sqrt{t}\, du$$

$$\boxed{\begin{array}{l} u = \sqrt{t} \\ du = \frac{1}{2\sqrt{t}}\,dt \end{array}}$$

$$= 200 \int_0^5 e^{u} \cdot u\, du = 200 \left[u e^{u} - e^{u} \right]_0^5$$

$$= 200\left(5e^5 - e^5\right) - 200(0-1) = 200 e^5 (4) + 200$$

$$= \underline{800 e^5 + 200}$$

Uttrykk for nåverdi: $\int_0^{25} f(t) e^{-rt}\,dt = \underline{\int_0^{25} 100\, e^{\sqrt{t}} e^{-rt}\,dt}$

4. a)

$$\left(\begin{array}{cccc|c} 2 & 1 & 2 & -3 & 4 \\ 3 & -1 & 8 & 2 & 7 \\ 5 & 5 & 0 & -17 & 12 \end{array}\right) -1 \rightarrow \left(\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 3 & -1 & 8 & 2 & 7 \\ 5 & 5 & 0 & -17 & 12 \end{array}\right) 3 \quad 5$$

$$\rightarrow \left(\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 0 & 5 & -10 & -13 & -2 \\ 0 & 15 & -30 & -42 & -3 \end{array}\right) -3 \rightarrow \left(\begin{array}{cccc|c} -1 & 2 & -6 & -5 & -3 \\ 0 & 5 & -10 & -13 & -2 \\ 0 & 0 & 0 & -3 & 3 \end{array}\right)$$

z fri

$-3w = 3 \qquad \underline{w = -1}$

$5y = 10z + 13(-1) - 2 = 10z - 15 \qquad y = \underline{2z - 3}$

$-x = -2(2z-3) + 6z + 5(-1) - 3 = 2z - 2 \qquad x = \underline{-2z+2}$

Uendelig mange løsn: $(x, y, z, w) = \underline{(-2z+2,\ 2z-3,\ z,\ -1)}$

med z fri

b)

$$\left(\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 3 & 1 & 2 & 4 \\ 2 & -1 & 4 & 3 \\ 4 & 5 & 1 & 13 \end{array}\right) -3 \; -2 \; -4 \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -8 & 11 & -2 \\ 0 & -7 & 10 & -1 \\ 0 & -7 & 13 & 5 \end{array}\right) -1 \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -7 & 10 & -1 \\ 0 & -7 & 13 & 5 \end{array}\right) -7 \; -7$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 6 & 12 \end{array}\right] \cdot -2 \quad \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 3 & 6 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

Én løsn.

$3z = 6 \quad \underline{z = 2}$

$-y = -(2) - 1 = -3 \quad y = \underline{3}$

$x = -3(3) + 3(2) + 2 = -1 \quad \underline{x = -1}$

Løsn:

$(x, y, z) = \underline{(-1, 3, 2)}$

5. a) $\begin{vmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{vmatrix} = a(a^2-9) - 2(2a-6) + 3(6-2a)$

$= a(a-3)(a+3) - 4(a-3) - 6(a-3)$

$= (a-3)(a(a+3) - 4 - 6)$

$= (a-3)(a^2+3a-10) = \underline{a^3 - 19a + 30}$

$= \underline{(a-3)(a-2)(a+5)}$

Ser at $a=3$ gjør at 2. og 3. rad er like, dvs $|A|=0$, så $a-3$ er faktor i $|A|$

b) $\underline{a=0}$: $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$ $\quad |A| = (-3)(-2)5 = \underline{30} \neq 0$

$\Rightarrow A^{-1}$ eksisterer

$$A^{-1} = \frac{1}{30}\begin{pmatrix} -9 & 6 & 6 \\ 9 & -6 & 4 \\ 6 & 6 & -4 \end{pmatrix}^T = \underline{\frac{1}{30}\begin{pmatrix} -9 & 9 & 6 \\ 6 & -6 & 6 \\ 6 & 4 & -4 \end{pmatrix}}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b} = \frac{1}{30}\begin{pmatrix} -9 & 9 & 6 \\ 6 & -6 & 6 \\ 6 & 4 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} -6 \\ -6 \\ 14 \end{pmatrix} = \underline{\begin{pmatrix} -1/5 \\ -1/5 \\ 7/15 \end{pmatrix}}$$

c) $A\underline{x} = \underline{b}$ har entydig løsn. $\Leftrightarrow |A| \neq 0$

$|A| = 0$: $a = 2, 3, -5$ $\Rightarrow$ Entydig løsn. for $\underline{a \neq 2, 3, -5}$

d) Mulige verdier av a med uendelig mange løsn.: $a = 2, 3, -5$

$a=2$:
$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Uendelig mange løsn.:

z fri $\quad y - z = -2 \Rightarrow y = z - 2$

$2x = -2(z-2) - 3z + 1 = -5z + 5 \Rightarrow x = -5z/2 + 5/2$

Løsn.: $(x, y, z) = \left(-5z/2 + 5/2,\ z-2,\ z\right)$ med z fri

$a=3$:
$$\left(\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 2 & 3 & 3 & 1 \\ 2 & 3 & 3 & -1 \end{array}\right) \Rightarrow \text{Ingen løsn.}$$

$a=-5$:
$$\left(\begin{array}{ccc|c} -5 & 2 & 3 & 1 \\ 2 & -5 & 3 & 1 \\ 2 & 3 & -5 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 2 & -5 & 3 & 1 \\ 2 & 3 & -5 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 0 & -21 & 21 & 7 \\ 0 & -13 & 13 & 5 \end{array}\right) \xrightarrow{-13/21} \left(\begin{array}{ccc|c} -1 & -8 & 9 & 3 \\ 0 & -21 & 21 & 7 \\ 0 & 0 & 0 & 5 - 13/3 \end{array}\right)$$

$5 - 13/3 \neq 0$

Ingen løsn.

6. a)
$$\left(\begin{array}{cc|c} 5 & 3 & 1 \\ 4 & 1 & 5 \\ 7 & 2 & 8 \end{array}\right) \longleftrightarrow x\underline{v}_1 + y\underline{v}_2 = \underline{v}_3$$

$$\to \left(\begin{array}{cc|c} 1 & 2 & -4 \\ 4 & 1 & 5 \\ 7 & 2 & 8 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -7 & 21 \\ 0 & -12 & 36 \end{array}\right) \xrightarrow{-12/7} \left(\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -7 & 21 \\ 0 & 0 & 0 \end{array}\right)$$

én løsn.

$$\left.\begin{array}{ll} -7y = 21 & y = -3 \\ x + 2(-3) = -4 & x = 6 - 4 = 2 \end{array}\right\} \quad \underline{v}_3 = 2\underline{v}_1 - 3\underline{v}_2$$

b) $\left(\begin{array}{cccc|c} 5 & 3 & 1 & 3 & a \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right)$ $]-1$ $\underline{w} = x\underline{v_1} + y\underline{v_2} + z\underline{v_3} + w\underline{v_4}$

$\to \left(\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right)$ $]-4$ $]-7$

$\to \left(\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & -12 & 36 & 48 & c-7(a-b) \end{array}\right)$ $]-12/7$

$\to \left(\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & -4a+5b \\ 0 & 0 & 0 & 0 & \underbrace{c-7(a-b)-\frac{12}{7}(b-4(a-b))}_{*=} \end{array}\right)$

$\underline{w}$ lin. kombinasjon av $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$

$\Leftrightarrow$ lin. system er konsistent

$\Leftrightarrow$ $\underline{* = 0}$: $c - 7(a-b) - \frac{12}{7}(b - 4(a-b)) = 0 \quad | \cdot 7$

$7c - 49(a-b) - 12b + 48(a-b) = 0$

$7c - 12b - (a-b) = 0$

$-a - 11b + 7c = 0 \quad | \cdot (-1)$

$\underline{a + 11b - 7c = 0}$

Konklusjon: $\underline{w}$ lin. komb. av $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ $\Leftrightarrow$ $\underline{a + 11b - 7c = 0}$

c) $\underline{w} \perp \underline{v_2} \Leftrightarrow \underline{w} \cdot \underline{v_2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 0 \Leftrightarrow \underline{3a + b + 2c = 0}$

b, c fri

$\frac{3a}{3} = \frac{-b-2c}{3}$

$a = -b/3 - 2c/3$

Løsn: $\underline{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b/3 - 2c/3 \\ b \\ c \end{pmatrix} = b/3 \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + c/3 \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$

$= \underline{s \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}}$

7. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$AX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

$XA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

<u>$AX = XA$:</u>

$\left.\begin{matrix} c = b \\ d = a \\ a = d \\ b = c \end{matrix}\right\}$ $\quad \begin{matrix} c, d \text{ fri} \\ a = d \\ b = c \end{matrix}$

<u>Konkl:</u>

$$\underline{X = c \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

med c, d fri

(alle lineær komb. av A og I)

$(a, b, c, d) = (d, c, c, d)$

$\underline{= c \cdot (0,1,1,0) + d(1,0,0,1)}$

c, d fri

8. a) $f'_x = 2x - 4y - 4 = 0$

$f'_y = -4x + 10y + 4 = 0$

$2x - 4y = 4$

$-4x + 10y = -4$

lin. sys.

$\left(\begin{matrix} 2 & -4 \\ -4 & 10 \end{matrix}\middle|\begin{matrix} 4 \\ -4 \end{matrix}\right) \Big]2$

$\sim$

$\left(\begin{matrix} 2 & -4 \\ 0 & 2 \end{matrix}\middle|\begin{matrix} 4 \\ 4 \end{matrix}\right)$

$2x - 4y = 4$

$2y = 4$

$2x = 4 \cdot 2 + 4 = 12$ $\quad \underline{x = 6}$

$\underline{y = 2}$

<u>Stasjonære pkt:</u> $(x, y) = \underline{(6, 2)}$

$H(f) = \begin{vmatrix} 2 & -4 \\ -4 & 10 \end{vmatrix}$ $\quad \det H(f) = 2 \cdot 10 - (-4)^2 = 20 - 16 = 4 > 0$

$\operatorname{tr} H(f) = 2 + 10 = 12 > 0$

$\Rightarrow (6,2)$ er <u>lokalt min</u> ved andrederivert-teste

$f(6,2) = 6^2 - 4 \cdot 6 \cdot 2 + 5 \cdot 2^2 - 4 \cdot 6 + 4 \cdot 2 + 1 = 36 - 48 + 20 - 24 + 8 + 1$

$= \underline{-7}$

b) Gjør variabelbytte $\left| \begin{array}{l} u = x-6 \\ v = y-2 \end{array} \right|$ slik at stasj. pkt. blir $u=v=0$. Dette gir $x = u+6$, $y = v+2$, og:

$$f(u,v) = (u+6)^2 - 4(u+6)(v+2) + 5(v+2)^2 - 4(u+6) + 4(v+2) + 1$$
$$= u^2 + 12u + 36 - 4(uv + 2u + 6v + 12) + 5(v^2 + 4v + 4)$$
$$- 4u - 24 + 4v + 8 + 1$$
$$= u^2 - 4uv + 5v^2 + \cancel{12u} - \cancel{8u} - \cancel{24v} + \cancel{20v} - \cancel{4u} + \cancel{4v}$$
$$+ 36 - 48 + 20 - 24 + 8 + 1$$
$$= u^2 - 4uv + 5v^2 - 7 = \underline{(u-2v)^2 + v^2 - 7} \geq -7$$

Dermed er $(u,v) = (0,0)$, eller $(x,y) = (6,2)$, globalt min pkt, og $f_{min} = \underline{-7}$.

Funksjonen f har ikke noe (lokalt eller globalt) maks.

9. a) $f'_x = 2xy^3 = 0$ $\quad$ $x=0$ eller $y=0$

$f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$ $\quad$ $2y-2=0$, $y=1$ $\;|\;$ $-2=0$ ingen pkt

$(x,y) = \underline{(0,1)}$

Stasjonære pkt:

$(x,y) = (0,1)$

$$H(f) = \begin{pmatrix} 2y^3 & 2x \cdot 3y^2 \\ 2x \cdot 3y^2 & x^2 \cdot 6y + 2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ det $H(f)(0,1) = 4 - 0 = 4 > 0$

tr — " — $= 2 + 2 = 4 > 0$

$\Rightarrow (x,y) = (0,1)$ er $\underline{\text{lokalt min}}$.

ved andrederivert-testen

b) Maksimum: $\underline{\text{Ingen}}$

Minimum: $f(0,1) = 1 - 2 + 1 = 0$ lokalt min.

$f(1,-3) = 1(-3)^3 + (-3)^2 - 2(-3) + 1 = -27 + 9 + 6 + 1 = -12 < 0$

$\Rightarrow$ $\underline{\text{Ingen minimum}}$ siden $f(0,1) = 0$ ikke er globalt min

(Faktisk har vi: $f(1,y) = y^3 + y^2 - 2y + 1 \to -\infty$ når $y \to -\infty$.)