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Research

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Research building on master thesis

In my master thesis, I studied connections on maximal Cohen-Macaulay modules (MCM modules) over simple curve singularities, and their monodromy. A central part of the thesis consisted of computations of (integrable) connections on the E_6 singularity, and their monodromy. The results obtained were later generalized and published in Eriksen [3]. I showed that any graded torsion free module over a plane irreducible quasi-homogeneous curve singularity admits a natural integrable connection. I also verified this result for the simple curve singularities that have several components.

Research building on doctoral thesis

In my doctoral thesis, Eriksen [0], I continued the study of differential structures on singularities, in particular on monomial curves. I considered differential structures such as the ring D of differential operators, D-modules, and modules with integrable connections, and I used noncommutative deformation theory and iterated extensions to classify graded holonomic D-modules on monomial curves.

Chapter 1-2 contains a detailed study of the ring D of differential operators on a monomial curve, which was later published in Eriksen [1]. I used the grading in an essential way, and obtained concrete algorithms for computing a generating set of homogeneous differential operators.

In *Chapter 3*, I used the results on the ring D and its associated graded ring to study the graded holonomic D-modules on a monomial curve. I defined new invariants using Hilbert functions to define and study holonomic D-modules, and classified all simple graded holonomic D-modules and their extensions. The results were later improved upon in the e-print Eriksen [6], which is submitted for publication.

In *Chapter 4*, I studied modules with (integrable) connections, the existence of which is a weaker requirement than existence of D-modules. This study is the basis for later works on connections on modules over isolated singularities, which I describe below in **Differential structures**. In the thesis, I considered the case of monomial curves. I showed that on a graded torsion-free module of rank one, there exists an integrable connection if the numerical semigroup defining the curve is symmetric. This generalizes the result in Eriksen [3] for plane quasi-homogeneous curve singularities of the form $x^p + y^q = 0$, since a numerical semigroup with two generators is symmetric. I also showed that given a graded torsion-free module of rank one over a monomial curve, it has D-module structures only if it is free.

In *Chapter 5*, I studied noncommutative deformations of modules. This is the basis for later work on noncommutative deformation theory, which I describe below in **Noncommutative Deformation Theory**. The thesis contains my version of noncommutative deformations of modules in the sense of Laudal, and an extended version was published in Eriksen [2]. It seems to me that my early accounts of noncommutative deformation theory have had a wide readership and been influential, as the papers by Laudal, with many proofs omitted, can be difficult to read for researchers that are new to the field.

In *Chapter 6*, I use iterated extensions and noncommutative deformation theory to classify all graded holonomic D-modules on monomial curves. I build upon the classification of the simple modules and their extensions mentioned above. The results on iterated extensions that I use have later been improved upon considerably in the e-print Eriksen [7], which is submitted for publication. In this e-print, I give an elementary and constructive proof of a criterion for a length category to be uniserial (which, it turns out, was known to Amdal and Ringdal already in 1968), and classify and explicitly construct all indecomposable objects in the uniserial case. We also show that all iterated extensions are completely determined by the noncommutative deformations of its simple factors.

Differential Structures

I have been interested in differential structures in algebraic geometry, and in particular on (isolated) singularities, since I worked with connections on modules over simple curve singularities in my master thesis, leading to the publication Eriksen [3] mentioned above. While working on my doctoral thesis, I became interested in the ring of differential operators and D-modules as well.

Later, Professor Trond S. Gustavsen (University of Bergen) shared my interest in differential structures. This led to a number of joint works on integrable connections on modules over isolated singularities. I describe the most important results and papers in this section.

In Eriksen, Gustavsen [8], we studied obstructions for the existence of (integrable) connections, and described and implemented an obstruction calculus. We developed a library connections.lib for computing these obstructions in Singular 3.0 (Greuel et al., 2005), a computer algebra system, and this library is available at `www.dr-eriksen.no/research/connections/`. As an application, we considered the existence of connections on maximal Cohen-Macaulay modules over isolated singularities. We verified several known results, in particular that all MCM modules over simple singularities in dimension one and two admit connections, and proved that for simple threefold singularities of type A_n, D_n, E_n, only the free MCM modules admit connections if $n \leq 50$. This, together with some scattered results in dimension $d = 4$, made us conjecture that for all simple singularities of dimension $d \geq 3$, an MCM module admits a connection if and only if it is free.

In Eriksen, Gustavsen [9], we proved this conjecture; an MCM module over a simple curve singularity of dimension $d \geq 3$ admits a connection if and only if it is free. An extended version of this paper, that contains all computations, is available as an e-print (arXiv: math/0606638). We also investigate singularities of finite CM representation type that are not simple. We found examples of curves that have MCM modules that do not admit connections, and threefolds that have non-free MCM modules that do admit connections. In dimension one, we prove that all gradable MCM modules over a monomial curve singularity admit connections if and only if the singularity is Gorenstein. More generally, we prove that over a Gorenstein curve singularity or a $\mathbf{Q}$-Gorenstein singularity of dimension $d \geq 2$, the canonical module admits a connection. Based on results for several other singularities of finite CM representation type, we conjecture that the property that the canonical module admits connections characterizes Gorenstein respectively $\mathbf{Q}$-Gorenstein singularities. In Eriksen, Gustavsen [10], we describe some related results.

In Eriksen, Gustavsen [11], we studied integrable connections on modules. When a torsion free module of rank one with a (not necessarily integrable) connection is given, we described an obstruction theory for the existence of integrable connections, expressed in terms of Lie-Rinehart cohomology. We proved that there is a canoncial obstruction in the second cohomology group, and that if it vanishes, then the moduli space of integrable connections is given by the first cohomology group. We applied the result to monomial curve singularities and normal CM singularities of dimension $d \geq 2$. The most explicit results are obtained in the case of quasi-homogeneous surface singularities, where we compute the Lie-Rinehart cohomology completely.

In Eriksen, Gustavsen [12], we studied equivariant Lie-Rinehart cohomology on the quotient of a singularity by a finite group. If A is the algebra of a normal domain of dimension two with the action of a finite group G, and $A^G \subseteq A$ is a Galois extension, then we were able to relate the Lie-Rinehart cohomology over the quotient to the Lie-Rinehart cohomology over A explicitly. We use this to compute the invariant Lie-Rinehart cohomology on the quotient of a quasi-homogeneous surface singularity by a finite group.

In another direction, it is known that the ring of differential operators is not always Noetherian or finitely generated, and there are well-known counter-examples for singular varieties in characteristic zero and smooth varieties in positive characteristic. I have been interested in whether these rings are coherent. In the e-print Eriksen [5], which is submitted for publication, I prove that the (full) ring D of differential operators on a smooth, finitely generated and connected k-algebra is coherent when k is a field of characteristic $p > 0$. I also ask whether the ring of differential operators on the cubic cone $A = k[x, y, z]/(x^3 + y^3 + z^3)$ is coherent when k is a field of characteristic 0. This is the famous counter-example of Bernstein, Gelfand and Gelfand. The coherence of D would make it possible to work with coherent D-modules and do homological algebra, and coherent D-modules are the ones that correspond to linear systems of differential equations with polynomial coefficients.

Noncommutative Deformation Theory

A central theme in my research is deformation theory, and in particular noncommutative deformation theory. My work in noncommutative deformation theory started with the paper Eriksen [2] mentioned above, which contains my version of noncommutative deformations of modules over an associative ring.

In Eriksen [4], I studied (global) noncommutative deformations of pre-sheaves and sheaves of modules. This is a generalization of both the commutative deformations of these global structures, and the noncommuative deformations of modules over an associative algebra. I first considered the deformation functor of a family of pre-sheaves of modules on a small category, and showed that is has a natural obstruction theory with global Hochschild cohomology as its cohomology. An important feature of this obstruction theory, is that it can be computed in concrete terms in many interesting cases. I also considered the deformation functor of a family of quasi-coherent sheaves of modules on a ringed space $(X, \mathcal{A})$. I proved that if $(X, \mathcal{A}) = (X, \mathcal{O}_X)$ is a scheme over a field k, or if it is a D-scheme in the sense of Beilinson and Bernstein, then the forgetful functor from the category of quasi-coherent sheaves into the category of pre-sheaves on U induces an isomorphism of deformation functors for any good $\mathcal{A}$-affine open cover U of X. As an application, we compute the noncommutative deformations of a D-module on an elliptic curve. The concrete results are obtained thanks to the above results and the fact that (X, D_X) is a D-scheme. Noncommutative global deformations have recently been used to develop new geometric invariants in an influential paper by Donovan, Wemyss (Duke Math. J. 2016).

In the monograph Eriksen, Laudal, Siqveland [13], we give a systematic and comprehensive introduction to noncommutative deformation theory, and applications to physical models and moduli problems. In addition to the material on noncommutative deformations, we develop a noncommutative phase space functor. I have written Chapter 1-3 of this monograph, which contains the heart of the text.

In the e-print Eriksen, Siqveland [15], which is submitted for publication, we consider the Generalized Burnside Theorem in noncommutative deformation theory. We give a complete proof of the theorem, and generalize it to the case of an arbitrary field. Earlier, we gave an overview of the role of the Generalized Burnside Theorem in noncommutative deformation theory in Eriksen, Siqveland [14], and used it to construct geometric structures for noncommutative algebras.

Several other papers on noncommutative deformation theory and noncommutative algebraic varieties are work in progress.

Selected Scientific Papers

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[3] ———, *Connections on modules over quasi-homogeneous plane curves*, Comm. Algebra **36** (2008), no. 8, 3032–3041.

[4] ———, *Computing noncommutative deformations of presheaves and sheaves of modules*, Canad. J. Math. **62** (2010), no. 3, 520–542.

[5] ———, *Coherent rings of differential operators*, ArXiv e-prints **1003.5151** (2018).

[6] ———, *Graded Holonomic D-modules on Monomial Curves*, ArXiv e-prints **1803.04367** (2018).

[7] ———, *Iterated Extensions and Uniserial Length Categories*, ArXiv e-prints **1804.03405** (2018).

[8] E. Eriksen and T. S. Gustaven, *Computing obstructions for existence of connections on modules*, J. Symbolic Comput. **42** (2007), no. 3, 313–323.

[9] ———, *Connections on modules over singularities of finite CM representation type*, J. Pure Appl. Algebra **212** (2008), no. 7, 1561–1574.

[10] ———, *Connections on modules over singularities of finite and tame CM representation type*, Generalized Lie theory in mathematics, physics and beyond, Springer, Berlin, 2009, pp. 99–108.

[11] ———, *Lie-Rinehart cohomology and integrable connections on modules of rank one*, J. Algebra **322** (2009), no. 12, 4283–4294.

[12] ———, *Equivariant Lie-Rinehart cohomology*, Proc. Est. Acad. Sci. **59** (2010), no. 4, 294–300.

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[14] E. Eriksen and A. Siqveland, *Geometry of noncommutative algebras*, Algebra, geometry and mathematical physics, Banach Center Publ., vol. 93, Polish Acad. Sci. Inst. Math., Warsaw, 2011, pp. 69–82.

[15] ———, *On the Generalized Burnside Theorem*, ArXiv e-prints **1702.07645** (2017).