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★**Noncommutative deformation theory.**

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In classical deformation theory, one studies families of algebraic or geometric objects parameterized by commutative local artinian rings. For example, suppose k is a field, A is a commutative k-algebra, and M is an A-module. A deformation of M parameterized by a commutative local artinian k-algebra (R, m) with residue field k is an $(R-A)$-bimodule M_R, flat over R, whose fiber over m is M. One can then define the *classical deformation functor*,

$$\mathrm{Def}_M\colon \mathsf{l} \to \mathsf{Sets},$$

where l is the category of commutative local artinian k-algebras with residue field k and local homomorphisms, by letting $\mathrm{Def}_M(R)$ be the set of (equivalence classes of) deformations of M parameterized by R. This functor can be extended to the category $\widehat{\mathsf{l}}$ of complete local commutative noetherian k-algebras with residue field k to give the functor of formal deformations of M. If X is a scheme and corresponds to a closed point x in a fine moduli space $\mathfrak{X}$, then the functor of formal deformations of X gives information about $\mathfrak{X}$ in a complete local neighborhood of x. This relates deformation theory to the geometry of moduli spaces.

The classical theory has been generalized to study noncommutative deformations of commutative schemes [M. Artin, in *Representation theory and algebraic geometry (Waltham, MA, 1995)*, 1–19, London Math. Soc. Lecture Note Ser., 238, Cambridge Univ. Press, Cambridge, 1997; MR1477464] and deformations of abelian categories (the infinitesimal theory was studied in [W. Lowen and M. Van den Bergh, Adv. Math. **198** (2005), no. 1, 172–221; MR2183254; Trans. Amer. Math. Soc. **358** (2006), no. 12, 5441–5483; MR2238922], while the formal theory was studied in [M. Van den Bergh, in *Derived categories in algebraic geometry*, 319–344, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012; MR3050709]). Both of these theories consider families over *commutative* local artinian rings.

In the book under review, classical deformation theory is generalized in another direction: deformations of algebraic objects over Artin algebras that are not necessarily commutative are considered. To be more precise, we introduce notation from the book. We let $r \geq 1$ be an integer, we let k^r denote the sum of r copies of k (with the ith copy denoted by k_i), and we call an algebra R *r-pointed* if there is a diagram of algebras

$$k^r \to R \to k^r$$

whose composition is the identity. Morphisms of r-pointed algebras are algebra maps compatible with the diagram above, giving a category A_r. The authors' replacement for the category l from the classical theory is the full subcategory a_r of A_r consisting of artinian rings with exact r simple (right) modules. This category includes truncations of free algebras over k and upper triangular $(r \times r)$-matrices. The authors then define a deformation of a collection of r objects $\{X_1, \ldots, X_r\}$ in a k-linear abelian category C over an object R of a_r to be a flat left R-object X in C together with a family of r isomorphisms $\eta_i\colon k_i \otimes_R X \to X_i$. This leads to the notion of *noncommutative deformation* *functor*

$$\mathrm{Def}_{\{X_1,\dots,X_r\}}\colon \mathsf{a}_r \to \mathsf{Sets}$$

as well as to the notion of the functor of formal noncommutative deformations of $\{X_1,\dots,X_r\}$. One of the primary themes of the text under review is to give conditions under which a deformation functor has a pro-representing hull, and this is related to the existence of a well-behaved obstruction theory.

We now describe the contents of the book under review. In the first chapter, the classical deformation theory of modules and associative algebras is reviewed and several examples are worked out in great detail.

Chapter 2 gives a brief overview of some basic results in noncommutative algebra that are invoked later, including the Artin-Wedderburn theorem, basic results about the Jacobson radical, and Burnside's theorem. Once again, the chapter is well written with enough detail to be suitable for graduate students unfamiliar with these results.

The heart of the text, Chapter 3, covers noncommutative deformation theory. After laying out the basic definitions, obstruction theory is described. Next, the case of deformation of modules is worked out in great detail, including a concrete description of an obstruction theory, and a method for computing the pro-representing hull when it exists. The authors then turn their attention to deformations of modules with a group action, and finally to deformations of sheaves and presheaves. Their analysis parallels that of the case of modules, in that they give concrete descriptions of obstruction theories, as well as methods for computing pro-representing hulls. The material is illustrated using examples, which greatly clarifies the abstract general theory. The chapter then turns to the relationship between obstruction theory and Massey products, and to a generalization of the classical Burnside theorem described in Chapter 2, and finally to a description of the deformations of particularly well-behaved sets of objects $\{X_1,\dots,X_r\}$, called *swarms*, over path algebras of ordered quivers.

The theory developed in Chapter 3 is illustrated (and motivated) further in Chapter 6. In this chapter (in which everything is over an algebraically closed field k of characteristic zero), the authors revisit the classical problem of constructing a moduli space of orbits of invertible 3×3-matrices acting, by conjugation, on the set of all 3×3-matrices. It is well known that there is no fine or coarse moduli space for this problem. But the authors show that there is a noncommutative moduli space. More precisely, let $G = \mathrm{GL}_3(k)$ and let A be the affine coordinate ring of the space of 3×3 matrices. The authors define a family of three rational A-G-modules, $\{X_1, X_2, X_3\}$, and prove that the functor $\mathrm{Def}_{\{X_1,X_2,X_3\}}$ has a pro-representing hull with an algebraization $\mathfrak{h}$ whose simple objects correspond to the orbits.

The other chapters of the book, Chapters 4 and 5, are related to another theme in the text, that of finding noncommutative geometric models for physical theories. In particular, it is the authors' hope to study promising mathematical frameworks for quantum gravity. In order to model a physical theory, one needs both a space of states (which the authors model as a noncommutative moduli space, e.g. the set of simple modules over a (possibly) noncommutative algebra), as well as a differentiable structure on this space. To define the latter, the authors associate to a finitely generated k-algebra A, a k-algebra $\mathrm{Ph}^\infty(A)$ together with a special derivation, the Dirac derivation $\delta\colon \mathrm{Ph}^\infty(A) \to \mathrm{Ph}^\infty(A)$. They then define a *dynamical structure* to be a quotient of the form $\mathrm{Ph}^\infty(A)/\sigma$, where σ is a δ-stable ideal. The rest of Chapter 4 is devoted to interpreting mathematical structures from general relativity and quantum field theory in this new geometric framework.

In Chapter 5, the authors study the noncommutative deformations of the algebra $k[x_1,x_2,x_3]/(x_1,x_2,x_3)^2$. In particular, they show that the deformation functor has a versal family over a base space, M, which contains a component isomorphic to $\mathrm{Hilb}^2(\mathbb{A}^3)$.

The structures developed in Chapter 4 are then computed on M, which is considered a toy model for the universe after the Big Bang.

This is the only textbook written describing its subject matter, written by those who developed it. The book is well written, with many detailed examples, making it ideal for graduate students and mathematicians interested in the subject matter. The reviewer looks forward to future applications and developments in this important subject.

Adam Nyman