Connections and monodromy on modules

Eivind Eriksen
Department of Mathematics
University of Oslo

May 22, 2000

Abstract

Let k be an algebraically closed field of characteristic 0, A a Noetherian k-algebra and M a finitely generated A-module. In this paper, we investigate the possible D-module structures on M lifting the A-module structure.

Existence of regular $Der_k(A)$-connections on M (or covariant derivations on M) is a first obstruction for lifting the structure on M. Subsequently, we develop an obstruction theory for connections on modules. This theory is almost completely due to Laudal.

Existence of regular $Der_k(A)$-connections on M is the only obstruction for lifting the structure in some cases, and it is essential that A is regular for this to happen. In this work, we want to investigate cases where the situation is a bit more complicated, and we therefore consider the class of indecomposable, maximal Cohen-Macaulay modules over simple curve singularities. We show that for this class of modules, there exists regular $Der_k(A)$-connections if A is irreducible.

Furthermore, we define the notion of monodromy of a connection on a module in cases where the module is well-behaved and the field $k = \mathbf{C}$. From the class of IMCM-modules over irreducible simple curve singularities mentioned above, we construct a new class of modules which is well-behaved, and which inherits connections from the IMCM-modules. We show that there exist connections on the IMCM-modules which gives trivial monodromy, and give explicit expressions for these connections.

1 Introduction

We fix the following notation throughout this paper: Let k be an algebraically closed field of characteristic 0, A a commutative, Noetherian k-algebra, and M a finitely generated A-module.

Let $End_k(A)$ denote the ring of k-linear endomorphisms of A. The ring of k-linear differential operators on A, $D = D(A) = \text{Diff}_k(A)$, is a non-commutative sub-algebra of $End_k(A)$ containing A, with the following filtration:

$$D^0(A) \subseteq D^1(A) \subseteq \ldots \subseteq D^p(A) \subseteq \ldots$$

The ring $D(A) = \sum_{p \geq 0} D^p(A)$, and each $D^p(A)$ is given in the following way: Let $\mu : A \otimes_k A \to A$ be the multiplication map, with kernel $J = ker(\mu)$, and consider $End_k(A)$ as an $A \otimes_k A$-module, by $(a \otimes b)\phi = a \circ \phi \circ b$ for all $a, b \in A,\ \phi \in End_k(A)$. Then, for each $p \geq 0$, let $D^p(A) = \{\phi \in End_k(A) : J^{p+1}\phi = 0\}$. In particular, this gives $D^0(A) = A$, and $D^1(A) = A \oplus Der_k(A)$.

The A-module structure of M is a ring homomorphism $\rho_0 : A \to End_k(M)$. A D-module structure on M is a ring homomorpism $\rho : D \to End_k(M)$, and it is a lifting of the A-module structure if the maps commute. The main problem adressed by this paper, is the following: Given an A-module structure on M, when can this be lifted to a D-module structure on M? And if this can be done, which such liftings exist?

To approach this question, we assume that a lifting $\rho : D \to End_k(M)$ is given. Obviously, this lifting induces a map $\nabla : Der_k(A) \to End_k(M)$. Since ρ is a ring homomorphism, it is clear that ∇ is A-linear, and a k-Lie algebra homomorphism. Furthermore, it has the derivation property, i.e. $\nabla(\delta)(am) = a\nabla(\delta)(m) + \delta(a)m$ for all $a \in A,\ m \in M,\ \delta \in Der_k(A)$. Such a map is henceforth called a regular $Der_k(A)$-connection on M.

At this point, we are led to consider the sub-algebra $\Delta(A) \subseteq End_k(A)$ generated by $D^1(A)$ (i.e. by A and $Der_k(A)$). This is in general a sub-algebra of $D(A)$, and we are interested in the case when $\Delta(A) = D(A)$. Then, there is a simple 1-1 correspondence between D-module structures on M lifting the A-module structure and regular $Der_k(A)$-connections on M.

A natural question is then, when is $\Delta(A) = D(A)$? In the case when A is a finitely generated k-algebra and an integral domain, this question has been given some thought. Nakai's conjecture simply states that in this situation, $\Delta(A) = D(A)$ if and only if A is regular. In [13], McConnell and Robson have shown that if A is a finitely generated, regular domain, $\Delta(A) = D(A)$, so one implication is obvious.

For the other, however, the results are more scattered: For $dim(A) = 1$, the conjecture has been proved by Mount and Villamayor in [14]. Furthermore, Chamarie and Musson has proven a more general result in [5]: If A is a finitely generated k-algebra with $dim(A) \leq 1$, and $\Delta(A) = D(A)$, A is regular.

From this, it seems that the study of connections on M would tell us a lot about the D-module structures on M compatible with the A-module structure. Of this reason, we are interested in a general theory for connections.

In this paper, we start by giving an obstruction theory of connections following Laudal. A connection on M is a k-linear map $\nabla : M \to M \otimes_A \Omega_{A/k}$ with derivation property. We show that there exists an obstruction $c(M) \in Ext^1_A(M, M \otimes_A \Omega_{A/k})$, called the Kodaira-Spencer class of M, which is canonical and has the property that $c(M) = 0$ if and only if there exists a connection on M. We also show that the set of connections on M is a torsor over $Hom_A(M, M \otimes_A \Omega_{A/k})$ if the obstruction vanishes.

Let $\mathbf{g}$ be a k-Lie sub-algebra and an A sub-module of $Der_k(A)$. Then, we define a $\mathbf{g}$-connection on M to be an A-linear map $\nabla : \mathbf{g} \to End_k(M)$ with derivation property. We show that there exists a natural map $g : Der_k(A) \to Ext^1_A(M, M)$, called the Kodaira-Spencer map, with the following property: The kernel $\mathbf{V}$ of this map is the maximal k-linear subspace $\mathbf{g}$ of $Der_k(A)$ such that there exists a k-linear map $\nabla : \mathbf{g} \to End_k(M)$ with derivation property. Furthermore, there exists a canonical obstruction $lc(M) \in Ext^1_A(\mathbf{V}, End_A(M))$, with the property that $lc(M) = 0$ if and only if there exists a $\mathbf{V}$-connection on M. The set of all $\mathbf{V}$-connections on M is a torsor over $Hom_A(\mathbf{V}, End_A(M))$ if the obstruction vanishes.

Finally, we define the curvature of a $\mathbf{V}$-connection ∇ on M to be the A-linear map $R_\nabla : \mathbf{V} \wedge_A \mathbf{V} \to End_A(M)$ given by $R_\nabla(\delta \wedge \eta) = [\nabla_\delta, \nabla_\eta] - \nabla_{[\delta,\eta]}$ for all $\delta, \eta \in \mathbf{V}$. We say that a $\mathbf{V}$-connection is regular or integrable if the curvature $R_\nabla = 0$. Any connection on M gives rise to $Der_k(A)$-connection on M, so we define the curvature of a connection on M to be the curvature of the corresponding $Der_k(A)$-connection. We say that a connection on M is regular or integrable if its curvature $R_\nabla = 0$.

In the last part of the paper, we study connections on some well-known classes of modules. First we consider simple, irredusible curve singularities A and indecomposable maximal Cohen-Macaulay A-modules M, and show the following theorem by explicit calculations:

Theorem 1.1 *Let A be a simple, irreducible curve singularity and M be an indecomposable, maximal Cohen-Macaulay A-module. Then there exists a regular $Der_k(A)$-connection on M. Furthermore, there exists a regular connection on M if and only if M is free.*

We also introduce a new class of modules closely related to the one mentioned above: For every simple, irreducible curve singularity $A = k[[x, y]]/(f)$ we consider the finitely generated k-algebra $B = k[x, y]/(f)$, and for every indecomposable maximal Cohen-Macaulay A-module M, we associate a finitely generated B-module N by means of the matrix factorizations of [7]. Assming the field $k = \mathbf{C}$, we show that for this new class of modules, we can associate a monodromy operator to every regular $\mathbf{V}$-connection in a natural way. By explicit calculations, we prove the following theorem:

Theorem 1.2 *Let B be a finitely generated k-algebra and N be a finitely generated B-module of the type mentioned above. Then there exists a regular $Der_k(B)$-connection on N with trivial monodromy.*

2 Connections

Let $\Omega_{A/k}$ be the module of Kähler-differentials of A, and $d : A \to \Omega_{A/k}$ the universal derivation of A. A connection on M is a k-linear map $\nabla : M \to M \otimes_A \Omega_{A/k}$ which has the property that

$$\nabla(am) = a\nabla(m) + m \otimes da$$

for all $a \in A,\ m \in M$. This property is called the derivation property of ∇.

A Lie-Cartan pair is a pair $(\mathbf{g}, \rho)$, such that $\mathbf{g}$ is a k-Lie algebra and an A-module, and $\rho : \mathbf{g} \to Der_k(A)$ is a k-Lie algebra homomorphism and an A-module homomorphism. If $\mathbf{g} \subseteq Der_k(A)$ is an A sub-module and a k-Lie sub-algebra, it is clear that $(\mathbf{g}, id)$ is a Lie-Cartan pair. In this case, we define a $\mathbf{g}$-connection on M to be an A-linear map $\nabla : \mathbf{g} \to End_k(M)$, which has the property that

$$\nabla_\delta(am) = a\nabla_\delta(m) + \delta(a)m$$

for all $\delta \in \mathbf{g},\ a \in A,\ m \in M$. This property is also called the derivation property of ∇. A $\mathbf{g}$-connection on M is often referred to as a covariant derivation or a logarithmic connection.

We immediately see that every connection on M gives rise to a $Der_k(A)$-connection on M through the bijection between $Der_k(A)$ and $Hom_A(\Omega_{A/k}, A)$. The opposite is not true in general. However, it is easy to see that if A is regular, a $Der_k(A)$-connection on M induces a connection on M. This gives a natural 1-1 correspondence between connections on M and $Der_k(A)$-connections on M when A is regular.

Finally, we say that a $\mathbf{g}$-connection ∇ on M is regular or integrable if ∇ is a k-Lie algebra homomorphism. In general, we define the curvature of ∇ to be the A-linear map $R_\nabla : \mathbf{g} \wedge_A \mathbf{g} \to End_A(M)$, given by $R_\nabla(\delta \wedge \eta) = [\nabla_\delta, \nabla_\eta] - \nabla_{[\delta,\eta]}$. We see that $R_\nabla = 0$ if and only if ∇ is regular. Furthermore, we say that the curvature of a connection ∇ on M is the curvature of the corresponding $Der_k(A)$-connection on M. The connection ∇ is regular or integrable if the corresponding $Der_k(A)$-connection is regular.

3 Hochschild cohomology

We want to give an obstruction theory for connections, and for this purpose we need a suitable cohomology. Subsequently, we define the following Hochschild-cohomology:

Let Q be any A-bimodule. We define Hochschild-cohomology on A with values in Q as the cohomology of the following complex: Let $C^n(A, Q) = Hom_k(\otimes_k^n A, Q)$ for all $n \geq 0$, and let $d^n : C^n(A, Q) \to C^{n+1}(A, Q)$ be given by

$$\begin{aligned} d^n(\phi)(a_1 \otimes \ldots \otimes a_{n+1}) &= a_1 \phi(a_2 \otimes \ldots \otimes a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \phi(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ &+ (-1)^{n+1} \phi(a_1 \otimes \ldots \otimes a_n) a_{n+1} \end{aligned}$$

for all $\phi \in C^n(A, Q)$. We denote the n'th Hochschild cohomology $HH^n(A, Q)$, and in general we have the following exact sequence:

$$0 \to HH^0(A, Q) \to Q \xrightarrow{d^0} Der_k(A, Q) \to HH^1(A, Q) \to 0$$

We are particularly interested in the case when $Q = Hom_k(N, P)$, for two A-modules N, P. Then, we have an isomorphism $HH^1(A, Hom_k(N, P)) \cong Ext^1_A(N, P)$, and the above sequence takes the form

$$\begin{aligned} 0 \to Hom_A(N, P) &\to Hom_k(N, P) \xrightarrow{d^0} \\ Der_k(A, Hom_k(N, P)) &\xrightarrow{\rho} Ext^1_A(N, P) \to 0 \end{aligned} \tag{1}$$

It turns out that this exact sequence will provide us with the necessary tool to construct an obstruction theory for connections.

4 An obstruction-theory for connections

In this section, we will give necessary and sufficient conditions for existence of connections on M and $\mathbf{g}$-connections on M. When connections of either of these two kinds exist, we will also give an accurate description of all other connections of the same kind.

Let us first restrict our attention to connections on M. The obstruction theory in this case is really quite simple, and is given by the following proposition:

Proposition 4.1 *There exists an obstruction* $c(M) \in Ext^1_A(M, M \otimes_A \Omega_{A/k})$*, which is canonical and has the property that* $c(M) = 0$ *if and only if there exists a connection on* M*. This obstruction is called the Kodaira-Spencer class of* M*. Furthermore, the set of all connections on* M *is a torsor over* $Hom_A(M, M \otimes_A \Omega_{A/k})$ *if the obstruction vanishes.*

Proof: Let $c \in Der_k(A, Hom_k(M, M \otimes_A \Omega_{A/k}))$ be given by $c(a)(m) = m \otimes da$ for all $a \in A,\ m \in M$. Then $c(M) = \rho(c) \in Ext^1_A(M, M \otimes_A \Omega_{A/k})$, in the exact sequence (1) with $N = M$ and $P = M \otimes_A \Omega_{A/k}$. We see that there is a connection on M if and only if $c = d^0(\eta)$ for an $\eta \in Hom_k(M, M \otimes_A \Omega_{A/k})$, so the first part of the proposition follows. If $\nabla,\ \nabla'$ are two connections on M, the difference $\nabla - \nabla' \in Hom_A(M, M \otimes_A \Omega_{A/k})$. Also, if ∇ is a connection on M, and $P \in Hom_A(M, M \otimes_A \Omega_{A/k})$, then $\nabla + P$ is a connection on M. $\square$

If there exists a regular connection on M, it can be shown that M is a locally free A-module, see for instance [6]. For this reason, existence of connections on M is a rather strong condition on M. As an example, we mention that if A is a local domain of dimension 1, any A-module is locally free if and only if it is free. Consequently, there exists a regular connection on an A-module M if and only if M is free.

Next, we study **g**-connections on M. This is a bit more complicated, because we must first find a candidate for **g**, and then give an obstruction for existence of **g**-connections on M for that **g**. At the same time, it is more interesting, because existence of **g**-connections on M seems a much weaker condition on M than existence of connections on M.

It is easy to see that $F = Ext^1_A(M, -)$ and $G = Der_k(A, Hom_k(M, -))$ are covariant functors from the category of A-modules into itself, when $Hom_k(M, N)$ is considered an A-module by left multiplication. Furthermore, any $\delta \in Der_k(A)$ gives rise to an A-linear homomorphism $\phi(\delta) : M \otimes_A \Omega_{A/k} \to M$ in an obvious way. By writing up the last part of the exact sequence (1) for $N = M,\ P = M \otimes_A \Omega_{A/k}$ and for $N = M,\ P = M$, by using the functors F and G on $\phi(\delta)$, and by considering the obvious map $\sigma : Der_k(A) \to Der_k(A, End_k(M))$, we get the following commutative diagram:

$$\begin{array}{rcl} Der_k(A, Hom_k(M, M \otimes_A \Omega_{A/k})) & \xrightarrow{\rho} & Ext^1_A(M, M \otimes_A \Omega_{A/k}) \to 0 \\ \downarrow G(\phi(\delta)) & & F(\phi(\delta)) \downarrow \\ End_k(M) \xrightarrow{d^0} Der_k(A, End_k(M)) & \xrightarrow{\rho} & Ext^1_A(M, M) \to 0 \\ \uparrow \sigma & & \\ Der_k(A) & & \end{array}$$

There is a canonical Kodaira-Spencer map $g : Der_k(A) \to Ext^1_A(M, M)$ given by the Kodaira-Spencer class $c(M)$. This map is defined by $g(\delta) = F(\phi(\delta))(c(M))$, and its kernel $\ker(g) \subseteq Der_k(A)$ is denoted $\mathbf{V}$. For all $\delta \in Der_k(A)$, we have $G(\phi(\delta))(c) = \sigma(\delta)$, so $g = \rho \circ \sigma$. Since ρ and σ are A-module homomorphisms, g is also an A-module homomorphism.

Proposition 4.2 *The pair* $(\mathbf{V}, id)$ *is a Lie-Cartan pair.*

Proof: Obviously, $\mathbf{V}$ is an A sub-module of $Der_k(A)$. To see that it is a k-Lie sub-algebra as well, consider $\delta, \eta \in \mathbf{V}$. Then $\sigma(\delta), \sigma(\eta) \in im(d_0)$, so there exist $\phi, \psi \in End_k(M)$ with $d^0(\phi) = \sigma(\delta),\ d^0(\psi) = \sigma(\eta)$. An easy computation now shows that $d^0([\phi, \psi]) = -\sigma([\delta, \eta])$, and the result follows. $\square$

Now, the notion of a $\mathbf{V}$-connection on M is well defined. The next proposition tells us that existence of $\mathbf{V}$-connections on M is in fact the best we can hope for:

Proposition 4.3 *There exists a k-linear map $\nabla : \mathbf{V} \to End_k(M)$ with derivation property. Furthermore, $\mathbf{V} \subseteq Der_k(A)$ is the maximal k-linear sub-space with this property.*

Proof: Let $\delta \in Der_k(A)$. Then $g(\delta) = 0$ if and only if $\sigma(\delta) \in im(d^0)$, so $\delta \in \mathbf{V}$ if and only if there exists a $\phi \in End_k(M)$ with $d^0(\phi) = \sigma(\delta)$. But there exists a $\nabla_\delta \in End_k(M)$ with derivation property with respect to δ if and only if this last condition if fulfilled (put $\nabla_\delta = -\phi$). This gives the maximality of $\mathbf{V}$. To see that there exits a k-linear map $\nabla : \mathbf{V} \to End_k(M)$, choose any k-base for $\mathbf{V}$ and any $\nabla_\delta \in End_k(M)$ with derivation property with respect to δ for each δ in this base. $\square$

It is now clear that $\mathbf{V}$ is the best candidate for a $\mathbf{g} \subseteq Der_k(A)$, such that $(\mathbf{g}, id)$ is a Lie-Cartan pair and $\mathbf{g}$-connections on M exist. We have seen that a k-linear map $\nabla : \mathbf{V} \to End_k(M)$ with derivation property always exists, and the next proposition describes when $\mathbf{V}$-connections M can be constructed from such a map.

Proposition 4.4 *There exists an obstruction $lc(M) \in Ext^1_A(\mathbf{V}, End_A(M))$, which is canonical and has the property that $lc(M) = 0$ if and only if there exists a $\mathbf{V}$-connection on M. Furthermore, the set of all $\mathbf{V}$-connections on M is a torsor over $Hom_A(\mathbf{V}, End_A(M))$ if the obstruction vanishes.*

Proof: Let $\nabla : \mathbf{V} \to End_k(M)$ be any k-linear map with derivation property, and consider the element $l_\nabla \in Der_k(A, Hom_k(\mathbf{V}, End_A(M)))$ given in the following way: For each $a \in A$, $\delta \in \mathbf{V}$, let $l_\nabla(a)(\delta) = \nabla(a\delta) - a\nabla(\delta)$. In the exact sequence (1) with $N = \mathbf{V}$ and $P = End_A(M)$, this gives rise to the element $lc(M) = \rho(l_\nabla)$. Note that $lc(M)$ is independent of the choice of ∇, so $lc(M) \in Ext^1_A(\mathbf{V}, End_A(M))$ is a canonical element. It is easy to see that $lc(M) = 0$ if and only if there exists an $\eta \in Hom_k(\mathbf{V}, End_A(M))$ such that $d^0(\eta) = l_\nabla$. But this last condition is equivalent to existence of an $\mathbf{V}$-connection on M (put $\nabla' = \nabla + \eta$), so the first part follows. If ∇, ∇' are $\mathbf{V}$-connections on M, then $\nabla - \nabla' \in Hom_A(\mathbf{V}, End_A(M))$. If ∇ is a $\mathbf{V}$-connection on M, and $P \in Hom_A(\mathbf{V}, End_A(M))$, then $\nabla + P$ is a $\mathbf{V}$-connection on M. $\square$

Finally, we describe how the Kodaira-Spencer class and the Kodaira-Spencer map can be calculated: Assume that M has a finite free resolution of the form

$$0 \leftarrow M \overset{\rho}{\leftarrow} L_0 \overset{d_0}{\leftarrow} L_1 \leftarrow \dots$$

where each $L_k = A^{n_k}$ for a $n_k \geq 1$, and each d_k is a matrix (a^k_{ij}) with entries in A. Such resolutions always exist when M is a finitely generated A-module. Then consider the map $(\rho \otimes_A id) \circ (da^0_{ij}) : L_1 \to M \otimes_A \Omega_{A/k}$. This gives rise to an element in $Ext^1_A(M, M \otimes_A \Omega_{A/k})$, and it is an easy computation to check that this element is in fact $c(M)$. It is then immediate that if $\delta \in Der_k(A)$ is any derivation, $g(\delta) \in Ext^1_A(M, M)$ is the element corresponding to the map $\rho \circ (\delta(a^0_{ij})) : L_1 \to M$.

5 An example

Let $B = k[x,y]/(x^3 + y^4)$, and N be the B-module given by the following resolution:

$$0 \leftarrow N \overset{\rho}{\leftarrow} B^3 \overset{d_0}{\leftarrow} B^3 \overset{d_1}{\leftarrow} B^3 \overset{d_0}{\leftarrow} B^3 \leftarrow \dots$$

where

$$d_0 = \begin{pmatrix} xy & -x^2 & -y^3 \\ -x^2 & -y^3 & xy^2 \\ y^2 & -xy & x^2 \end{pmatrix}, \; d_1 = \begin{pmatrix} 0 & -x & y^2 \\ -x & -y & 0 \\ -y & 0 & x \end{pmatrix}$$

The module of derivations on B, $Der_k(B)$, is generated by the Euler-derivation δ_0 and the trivial derivation δ_1, which are given by:

$$\delta_0 = 4x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} \quad , \quad \delta_1 = 4y^3\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y}$$

We have the following relations in $Der_k(B)$: $x^2\delta_0 + y\delta_1 = 0$ and $y^3\delta_0 - x\delta_1 = 0$. Finally, $Der_k(B)$ is a k-Lie algebra, with $[\delta_0, \delta_1] = 7\delta_1$.

If there exists a regular connection on N, N is a locally free B-module. It is easy to see that this is not the case, so regular connections on N cannot exist.

To calculate the Kodaira-Spencer kernel $\mathbf{V}$ of N, it is enough to find $g(\delta_0)$ and $g(\delta_1)$. As elements in $Hom_B(B^3, N)$, we have the following equalities:

$$g(\delta_0) = \begin{pmatrix} 7xy & -8x^2 & -9y^3 \\ -8x^2 & -9y^3 & 10xy^2 \\ 6y^2 & -7xy & 8x^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \circ d_0$$

$$g(\delta_1) = \begin{pmatrix} -7x^3 & -8xy^3 & 9x^2y^2 \\ -8xy^3 & 9x^2y^2 & -10x^3y \\ -6x^2y & 7x^3 & 8xy^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -xy \\ x & 0 & 0 \end{pmatrix} \circ d_0$$

Hence, we get $g(\delta_0) = g(\delta_1) = 0$ in $Ext^1_B(N, N)$, so $g = 0$ and the Kodaira-Spencer kernel $\mathbf{V} = Der_k(B)$. Furthermore, we have a k-linear map $\nabla : \mathbf{V} \to End_k(N)$ with derivation property: Consider the natural k-base for $B\delta_0$ as k-vectorspace, and add δ_1 to obtain k-base for $\mathbf{V}$. Then let ∇_{δ_0} and ∇_{δ_1} be given by

$$\nabla_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & xy \\ -x & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and make a k-linear extension to obtain ∇.

Next, we calculate the obstruction $lc(N) \in Ext^1_B(\mathbf{V}, End_B(N))$. This obstruction is 0 if and only if there exists an element $\eta \in Hom_k(\mathbf{V}, End_B(N))$ such that $\nabla_{b\delta} - b\nabla_\delta = b\eta(\delta) - \eta(b\delta)$ for all $b \in B$, $\delta \in \mathbf{V}$. We define elements $P_{n,m} = \nabla_{x^ny^m\delta_1} - x^ny^m\nabla_{\delta_1} \in End_B(N)$ for all $n, m \geq 0$. By identifying p_i with $\eta(\delta_i)$ for $i = 1, 2$, we see that $lc(N) = 0$ if and only there exists $p_0, p_1 \in End_B(N)$ which satisfy the following equations in $End_B(N)$:

$$\begin{aligned} -x^2p_0 - yp_1 &= P_{1,0} \\ y^3p_0 - xp_1 &= P_{0,1} \end{aligned}$$

It turns out that there exists a simple solution to these equations, obtained by putting $p_0 = 0$. As elements in $End_B(N)$, we have:

$$P_{1,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -x^2 & xy^2 \\ -xy & 0 & x^2 \end{pmatrix} = \begin{pmatrix} 0 & -xy & y^3 \\ 0 & 0 & 0 \\ -xy & -y^2 & 0 \end{pmatrix}$$

$$P_{0,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^3 & x^2y \\ -x^2 & 0 & -y^3 \end{pmatrix} = \begin{pmatrix} 0 & -x^2 & xy^2 \\ 0 & 0 & 0 \\ -x^2 & -xy & 0 \end{pmatrix}$$

In this way, we get the following solution to the system:

$$p_0 = 0 \ , \ p_1 = \begin{pmatrix} 0 & x & -y^2 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix}$$

These calculations show that the obstruction $lc(N) = 0$, and that there exist a $\mathbf{V}$-connection ∇^0 on N, given by $\nabla^0_{\delta_0} = \nabla_{\delta_0} + p_0$, $\nabla^0_{\delta_1} = \nabla_{\delta_1} + p_1$. Explicitly, this connection takes the following form:

$$\nabla^0_{\delta_0} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^0_{\delta_1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & x & -y^2 \\ 0 & 0 & xy \\ 0 & y & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

A straight-forward computation shows that this connection is regular.

Assume that the field $k = \mathbf{C}$, the complex numbers. In this case, there exists a natural monodromy operator associated to the $\mathbf{V}$-connection ∇^0 on N: Consider the open, affine subset $D(x) \subseteq Spec(B)$. This subset contains all the closed, non-singular points of $Spec(B)$, and it is isomorphic to the ring $Spec(C)$ for $C = B_{\{x\}}$. But $C \cong k[t, t^{-1}]$, so there is a natural 1-1 correspondance between the closed points of $D(x)$ and the points of $\mathbf{C}^*$. It is easy to see that $N_{\{x\}}$ is C-free of rank 2, so there exist an isomorphism $\phi : N_{\{x\}} \to C^2$. When this isomorphism is fixed, each $Der_k(B)$-connection on N induces a $Der_k(C)$-connection on $N_{\{x\}}$. But $Der_k(C)$ is generated by the derivation $\frac{d}{dt}$, so a $Der_k(C)$-connection on $N_{\{x\}}$ is exactly the same as an action on $k[t, t^{-1}]^2$ with derivation property with respect to $\frac{d}{dt}$. Let $\phi : N_{\{x\}} \to C^2$ be the isomorphism defined by

$$\phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + tc \\ b + t^2 c \end{pmatrix}$$

Then the $Der_k(B)$-connection ∇^0 on N induces the following action on $k[t, t^{-1}]^2$:

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} \frac{df}{dt} \\ \frac{dg}{dt} - \frac{g}{t} \end{pmatrix}$$

When we identify C with the functions on $\mathbf{C}^*$, such an action gives a system of 2 linear, first-order differential equations defined on $\mathbf{C}^*$. In our case, we get the following system:

$$\begin{aligned} \frac{df}{dt} &= 0 \\ \frac{dg}{dt} &= \frac{g}{t} \end{aligned}$$

In a neighborhood of each point in $\mathbf{C}^*$, the solutions of this system is a 2-dimensional k-vectorspace. In our case, a basis for this vectorspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix} \right\}$$

Notice that these solutions are in fact global solutions.

The monodromy operator associated to the $Der_k(B)$-connection ∇^0 on N is the monodromy operator associated with this system of differential equations. It is easy to see that this monodromy operator is independent of the choice of isomorphism $\phi : N_{\{x\}} \to C^2$, so this monodromy operator is in fact well-defined. In our case, we have 2 linearly independent, global solutions, so the monodromy operator is trivial.

6 Simple curve singularities

Let (R, m, k) be a local Noetherian k-algebra, and P be a finitely generated R-module. A sequence $\{x_1, \ldots, x_n\}$ of elements in m is called P-regular if the following condition hold: For every i with $0 \leq i < n$, $x_{i+1}p = 0 \Rightarrow p = 0$ for every $p \in P/(x_1, \ldots, x_i)P$. The maximal length of P-regular sequences, which is a well-defined integer by [12], is called the depth of P. The R-module P is maximal Cohen-Macaulay (or MCM) if $depth(P) = dim(R)$.

We may now return to our original setting, where A is a Noetherian k-algebra, and M is a finitely generated A-module. We say that M is a MCM A-module if M_p is a MCM A_p-module for every prime ideal p in A. Furthermore, A is of finite CM representation type if the number of isomorphism classes of indecomposable MCM (or IMCM) A-modules is finite.

Consider the complete, local k-algebra $R = S/(f)$, where $S = k[[z_1, \ldots, z_n]]$ and $f \in S$ is a non-zero, non-unit element, and define

$$c(f) = \{I : I \text{ is a proper ideal of } S \text{ with } f \in I^2\}$$

We say that R is a simple hypersurface singularity if the set $c(f)$ is finite, see [16]. It has been shown by [10] and [4] that R is a simple hypersurface singularity if and only if R is of finite CM representation type.

We are interested in one-dimentional simple hypersurface singularities, and we will from now on refer to these as simple curve singularities. They have been classified by [1], and have the form $A = k[[x, y]]/(f)$, where $f \in k[[x, y]]$ is a polynomial from the following, well-known list:

Name:	f:	
A_n	$x^2 + y^{n+1}$	n=1,2,...
D_n	$x^2y + y^{n-1}$	n=4,5,...
E_6	$x^3 + y^4$	
E_7	$x^3 + xy^3$	
E_8	$x^3 + y^5$	

We introduce some notation: If A is irreducible, let K denote the quotient field of A and define the rank of M as $rk(M) = dim_K(M \otimes_A K)$. We say that M is a torsion free A-module if $Ann(m) = 0$ for all $m \neq 0$ in M. If A is a simple curve singularity, it is well-known that M is torsion free if and only if M is MCM.

Finally, we introduce the important notion of matrix factorizations following Eisenbud: Let S be a Noetherian k-algebra, and $f \in S$ a non-zero, non-unit element. A matrix factorization of f over S is a pair of morphisms (ϕ, ψ), where $\phi : S^n \to S^m$ and $\psi : S^m \to S^n$, such that $\phi \circ \psi = f \circ id_{S^m}$ and $\psi \circ \phi = f \circ id_{S^n}$. Let $R = S/(f)$, and consider the complex

$$R^n \overset{\overline{\phi}}{\leftarrow} R^m \overset{\overline{\psi}}{\leftarrow} R^n \leftarrow \ldots$$

induced by the matrix factorization. If $(f)/(f^2)$ is R-free, Eisenbud showed in [7] that this a free resolution of the finitely generated R-module $coker(\phi)$ and we have $n = m$.

Let S be a regular, local Noetherian k-algebra, $f \in S$ a non-zero, non-unit element and $R = S/(f)$. If P is a finitely generated R-module with a minimal R-free resolution (L_i, d_i), Eisenbud showed in [7] that (L_i, d_i) is induced by a matrix factorization of f over S if and only if P is an MCM R-module with no free summand.

Let us consider a simple curve singulartity $A = k[[x,y]]/(f)$. We know that A is of finite CM representation type, so there are only finitely many isomorphism classes of IMCM A-modules. These modules have been classified by [8]. They all have free resolutions which are induced by matrix factorizations of f over $k[[x,y]]$. Because of the simplicity of this class of modules, it is a natural testing ground for the theory of connections. We proceed by making some explicit calculations for these modules.

7 Calculations

In this section, we give explicit expressions for every IMCM A-module M, in terms of free resolutions induced by matrix factorizations, when A is a simple, irreducible curve singularity. For each of these A-modules, we give an explicit expression for one regular $Der_k(A)$-connection on M.

7.1 The singularity A_n when n is even

We consider the singularity A_n for n even, so let $n = 2p$ with $p \geq 1$, and let $f = x^2 + y^{2p+1}$. In this case, the derivation module $Der_k(A)$ is generated by the Euler-derivation δ_0 and the trivial derivation δ_1:

$$\delta_0 = (2p+1)x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \quad , \quad \delta_1 = (2p+1)y^{2p}\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}$$

We have the following relations in $Der_k(A)$: $y^{2p}\delta_0 - x\delta_1 = 0$ and $x\delta_0 + y\delta_1 = 0$.

There are $p+1$ IMCM A-modules M, and we refer to them as $M_0, \dots, M_p$. The A-module M_i has free resolution

$$0 \leftarrow M_i \leftarrow A^n \xleftarrow{d_{0,i}} A^n \xleftarrow{d_{1,i}} A^n \leftarrow \dots$$

for $0 \leq i \leq p$, where the differentials are given by:

$0 \leq i \leq p$: $rk(M_i) = 1$

$$d_{0,0} = \left(x^2 + y^{2p+1}\right) \qquad\qquad d_{1,0} = (1)$$

$$d_{0,i} = \begin{pmatrix} x & (-y)^{p+i} \\ -(-y)^{p-i+1} & -x \end{pmatrix} \quad d_{1,i} = \begin{pmatrix} x & (-y)^{p+i} \\ -(-y)^{p-i+1} & -x \end{pmatrix}$$

for $1 \leq i \leq p$.

The module M_0 is free of rank 1, so there is a trivial $Der_k(A)$-connection on M_0. Furthermore, there is a $Der_k(A)$-connection ∇^i on M_i for $1 \leq i \leq p$, given by:

$$\begin{aligned} \nabla^i_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2i-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ \nabla^i_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & (2i-1)(-y)^{p+i-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

Explicit calculations show that all these connections are regular.

7.2 The singularity E_6

We consider the singularity E_6, so let $f = x^3 + y^4$. In this case, the derivation module $Der_k(A)$ is generated by the Euler-derivation δ_0 and the trivial derivation δ_1:

$$\delta_0 = 4x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} \quad , \quad \delta_1 = 4y^3\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y}$$

We have the following relations in $Der_k(A)$: $x^2\delta_0 + y\delta_1 = 0$ and $y^3\delta_0 - x\delta_1 = 0$.

There are 7 IMCM A-modules M, and we refer to them as $M_1, \dots, M_7$. The A-module M_i has free resolution

$$0 \leftarrow M_i \leftarrow A^n \overset{d_{0,i}}{\leftarrow} A^n \overset{d_{1,i}}{\leftarrow} A^n \leftarrow \dots$$

for $1 \leq i \leq 7$, where the differentials are given by:

$1 \leq i \leq 5$: $rk(M_i) = 1$

$$d_{0,1} = \left(x^3 + y^4\right) \qquad d_{1,1} = (1)$$

$$d_{0,2} = \begin{pmatrix} y & -x^2 \\ x & y^3 \end{pmatrix} \qquad d_{1,2} = \begin{pmatrix} y^3 & x^2 \\ -x & y \end{pmatrix}$$

$$d_{0,3} = \begin{pmatrix} y^2 & -x^2 \\ x & y^2 \end{pmatrix} \qquad d_{1,3} = \begin{pmatrix} y^2 & x^2 \\ -x & y^2 \end{pmatrix}$$

$$d_{0,4} = \begin{pmatrix} y^3 & -x^2 \\ x & y \end{pmatrix} \qquad d_{1,4} = \begin{pmatrix} y & x^2 \\ -x & y^3 \end{pmatrix}$$

$$d_{0,5} = \begin{pmatrix} x & -y^2 & 0 \\ y & 0 & -x \\ 0 & -x & -y \end{pmatrix} \quad d_{1,5} = \begin{pmatrix} x^2 & y^3 & -xy^2 \\ -y^2 & xy & -x^2 \\ xy & -x^2 & -y^3 \end{pmatrix}$$

$6 \leq i \leq 7$: $rk(M_i) = 2$

$$d_{0,6} = \begin{pmatrix} xy & -x^2 & -y^3 \\ -x^2 & -y^3 & xy^2 \\ y^2 & -xy & x^2 \end{pmatrix} \qquad d_{1,6} = \begin{pmatrix} 0 & -x & y^2 \\ -x & -y & 0 \\ -y & 0 & x \end{pmatrix}$$

$$d_{0,7} = \begin{pmatrix} xy & -x^2 & -y^2 & 0 \\ -x^2 & -y^3 & 0 & y^2 \\ y^2 & -xy & 0 & x \\ 0 & 0 & -x & y \end{pmatrix} \quad d_{1,7} = \begin{pmatrix} 0 & -x & y^2 & 0 \\ -x & -y & 0 & y^2 \\ -y^2 & 0 & xy & -x^2 \\ -xy & 0 & x^2 & y^3 \end{pmatrix}$$

The module M_1 is free of rank 1, so there is a trivial $Der_k(A)$-connection on M_1. Furthermore, there is a $Der_k(A)$-connection ∇^i on M_i for $2 \leq i \leq 7$, given by:

$$\nabla^2_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^2_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^3_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^3_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & 2xy \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^4_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^4_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & 5xy^2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^5_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^5_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & -y^2 & 2xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^6_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^6_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & x & -y^2 \\ 0 & 0 & xy \\ 0 & y & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^7_{\delta_0}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \\ \delta_0(d) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\nabla^7_{\delta_1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \\ \delta_1(d) \end{pmatrix} + \begin{pmatrix} 0 & x & 0 & 2xy \\ 0 & 0 & xy & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Explicit calculations show that all these connections are regular.

7.3 The singularity E_8

Consider the singularity E_8, so let $f = x^3 + y^5$. In this case, the derivation module $Der_k(A)$ is generated by the Euler-derivation δ_0 and the trivial derivation δ_1:

$$\delta_0 = 5x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} \quad , \quad \delta_1 = 5y^4\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y}$$

We have the following relations in $Der_k(A)$: $y^4\delta_0 - x\delta_1 = 0$ and $x^2\delta_0 + y\delta_1 = 0$.

There are 17 IMCM A-modules M, and we refer to them as $M_1, \dots, M_{17}$. The A-module M_i has the following free resolution

$$0 \leftarrow M_i \leftarrow A^n \overset{d_{0,i}}{\leftarrow} A^n \overset{d_{1,i}}{\leftarrow} A^n \leftarrow \dots$$

for $1 \leq i \leq 17$, where the differentials are given by:

$1 \leq i \leq 7$: $rk(M_i) = 1$

$$d_{0,1} = \left(x^3 + y^5\right) \qquad d_{1,1} = (1)$$

$$d_{0,2} = \begin{pmatrix} x & y^3 \\ -y^2 & x^2 \end{pmatrix} \qquad d_{1,2} = \begin{pmatrix} x^2 & -y^3 \\ y^2 & x \end{pmatrix}$$

$$d_{0,3} = \begin{pmatrix} x & -y^4 \\ y & x^2 \end{pmatrix} \qquad d_{1,3} = \begin{pmatrix} x^2 & y^4 \\ -y & x \end{pmatrix}$$

$$d_{0,4} = \begin{pmatrix} x & -y^2 \\ y^3 & x^2 \end{pmatrix} \qquad d_{1,4} = \begin{pmatrix} x^2 & y^2 \\ -y^3 & x \end{pmatrix}$$

$$d_{0,5} = \begin{pmatrix} x & y \\ -y^4 & x^2 \end{pmatrix} \qquad d_{1,5} = \begin{pmatrix} x^2 & -y \\ y^4 & x \end{pmatrix}$$

$$d_{0,6} = \begin{pmatrix} x & y^3 & 0 \\ y & 0 & x \\ 0 & x & y \end{pmatrix} \quad d_{1,6} = \begin{pmatrix} x^2 & y^4 & -xy^3 \\ y^2 & -xy & x^2 \\ -xy & x^2 & y^4 \end{pmatrix}$$

$$d_{0,7} = \begin{pmatrix} x & y^2 & 0 \\ 0 & -x & y^2 \\ y & 0 & -x \end{pmatrix} \quad d_{1,7} = \begin{pmatrix} x^2 & xy^2 & y^4 \\ y^3 & -x^2 & -xy^2 \\ xy & y^3 & -x^2 \end{pmatrix}$$

$8 \leq i \leq 14$: $rk(M_i) = 2$

$$d_{0,8} = \begin{pmatrix} x^2 & -y^3 & xy^2 \\ xy^2 & x^2 & y^4 \\ y^2 & xy & -x^2 \end{pmatrix} \qquad d_{1,8} = \begin{pmatrix} x & 0 & y^2 \\ -y^2 & x & 0 \\ 0 & y & -x \end{pmatrix}$$

$$d_{0,9} = \begin{pmatrix} -y^3 & xy^2 & x & 0 \\ x^2 & y^4 & 0 & xy \\ xy & -x^2 & 0 & y^2 \\ 0 & 0 & y & -x \end{pmatrix} \quad d_{1,9} = \begin{pmatrix} -y^2 & x & 0 & xy \\ 0 & y & -x & 0 \\ x^2 & 0 & xy^2 & y^4 \\ xy & 0y^3 & -x^2 & \end{pmatrix}$$

$$d_{0,10} = \begin{pmatrix} -y^3 & x^2 & 0 & -y^2 \\ x^2 & xy^2 & -y^3 & 0 \\ xy & y^3 & 0 & x \\ 0 & 0 & -x & y \end{pmatrix} \quad d_{1,10} = \begin{pmatrix} -y^2 & x & 0 & -y^3 \\ x & 0 & y^2 & 0 \\ 0 & -y^2 & xy & -x^2 \end{pmatrix}$$

$$d_{0,11} = \begin{pmatrix} -y^3 & x & 0 & 0 & -y^2 \\ x^2 & 0 & xy & -y^3 & 0 \\ xy & 0 & y^2 & 0 & x \\ 0 & y & -x & 0 & 0 \\ 0 & 0 & 0 & -x & y \end{pmatrix}$$

$$d_{1,11} = \begin{pmatrix} -y^2 & x & 0 & xy & -y^3 \\ x^2 & 0 & xy^2 & y^4 & 0 \\ xy & 0 & y^3 & -x^2 & 0 \\ 0 & -y^2 & xy & 0 & -x^2 \\ 0 & -xy & x^2 & 0 & y^4 \end{pmatrix}$$

$$d_{0,12} = \begin{pmatrix} -xy & x^2 & y^4 \\ x^2 & y^4 & -xy^3 \\ -y^2 & xy & -x^2 \end{pmatrix} \qquad d_{1,12} = \begin{pmatrix} 0 & x & -y^3 \\ x & y & 0 \\ y & 0 & -x \end{pmatrix}$$

$$d_{0,13} = \begin{pmatrix} -xy & x^2 & y^2 & 0 \\ x^2 & y^4 & 0 & y^3 \\ -y^2 & xy & 0 & x \\ 0 & 0 & -x & -y^2 \end{pmatrix} \quad d_{1,13} = \begin{pmatrix} 0 & x & -y^3 & 0 \\ x & y & 0 & y^2 \\ y^3 & 0 & -xy^2 & -x^2 \\ -xy & 0 & x^2 & -y^3 \end{pmatrix}$$

$$d_{0,14} = \begin{pmatrix} -xy & x^2 & -y^3 & 0 \\ x^2 & y^4 & 0 & y^3 \\ -y^2 & xy & 0 & x \\ 0 & 0 & -x & y \end{pmatrix} \quad d_{1,14} = \begin{pmatrix} 0 & x & -y^3 & 0 \\ x & y & 0 & -y^3 \\ -y^2 & 0 & xy & -x^2 \\ -xy & 0 & x^2 & y^4 \end{pmatrix}$$

$15 \leq i \leq 17$**:** $rk(M_i) = 3$

$$d_{0,15} = \begin{pmatrix} -y^3 & x^2 & xy^2 & -xy & 0 \\ x^2 & xy^2 & y^4 & 0 & xy \\ 0 & 0 & 0 & x^2 & -y^3 \\ xy & y^3 & -x^2 & 0 & y^2 \\ 0 & 0 & 0 & -y^2 & -x \end{pmatrix}$$

$$d_{1,15} = \begin{pmatrix} -y^2 & x & 0 & 0 & xy \\ x & 0 & y & y^2 & 0 \\ 0 & y & 0 & -x & 0 \\ 0 & 0 & x & 0 & -y^3 \\ 0 & 0 & -y^2 & 0 & -x^2 \end{pmatrix}$$

$$d_{0,16} = \begin{pmatrix} -y^3 & x^2 & -xy & 0 & 0 & -y^2 \\ x^2 & xy^2 & 0 & xy & -y^3 & 0 \\ 0 & 0 & x^2 & -y^3 & 0 & 0 \\ xy & y^3 & 0 & y^2 & 0 & x \\ 0 & 0 & -y^2 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & y \end{pmatrix}$$

$$d_{1,16} = \begin{pmatrix} -y^2 & x & 0 & 0 & xy & -y^3 \\ x & 0 & y & y^2 & 0 & 0 \\ 0 & 0 & x & 0 & -y^3 & 0 \\ 0 & 0 & -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & 0 & xy & 0 & -x^2 \\ 0 & -xy & 0 & x^2 & 0 & y^4 \end{pmatrix}$$

$$d_{0,17} = \begin{pmatrix} -y^3 & x^2 & xy^2 & 0 & -y^2 & 0 \\ 0 & 0 & 0 & y^2 & x & 0 \\ x^2 & xy^2 & y^4 & 0 & 0 & -y^3 \\ xy & y^3 & -x^2 & 0 & x & 0 \\ 0 & 0 & 0 & -x & 0 & y^2 \\ 0 & 0 & 0 & 0 & y & -x \end{pmatrix}$$

$$d_{1,17} = \begin{pmatrix} -y^2 & 0 & x & 0 & 0 & -y^3 \\ x & 0 & 0 & y^2 & 0 & 0 \\ 0 & x & y & -x & y^2 & 0 \\ 0 & y^3 & 0 & 0 & -x^2 & -xy^2 \\ 0 & x^2 & 0 & 0 & xy^2 & y^4 \\ 0 & xy & 0 & 0 & y^3 & -x^2 \end{pmatrix}$$

The module M_1 is free of rank 1, so there is a trivial $Der_k(A)$-connection on M_1. Furthermore, there is a $Der_k(A)$-connection ∇^i on M_i for $2 \leq i \leq 17$, given by:

$$\begin{aligned}
\nabla^2_{\delta_0} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
\nabla^2_{\delta_1} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & -y^2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
\nabla^3_{\delta_0} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
\nabla^3_{\delta_1} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & -2y^3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

$$\nabla^4_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^4_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & 4y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^5_{\delta_0}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^5_{\delta_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \end{pmatrix} + \begin{pmatrix} 0 & -7 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\nabla^6_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^6_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & -2y^3 & 0 \\ 0 & 0 & -4y^3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^7_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^7_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -2y^3 \\ 0 & 0 & 0 \\ 0 & y^2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^8_{\delta_0}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^8_{\delta_1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \end{pmatrix} + \begin{pmatrix} 0 & y^2 & -xy \\ 0 & 0 & -y^3 \\ 0 & -x & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\nabla^9_{\delta_0}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \\ \delta_0(d) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\nabla^9_{\delta_1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \\ \delta_1(d) \end{pmatrix} + \begin{pmatrix} 0 & y^2 & 0 & -2y^3 \\ 0 & 0 & -y^3 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & y^2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\nabla^{10}_{\delta_0}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \\ \delta_0(d) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\nabla^{10}_{\delta_1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \delta_1(a) \\ \delta_1(b) \\ \delta_1(c) \\ \delta_1(d) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -xy & -3x^2 \\ 0 & 0 & -y^3 & 0 \\ 0 & 0 & 0 & -3y^3 \\ 0 & y & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\nabla^{11}_{\delta_0}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} \delta_0(a) \\ \delta_0(b) \\ \delta_0(c) \\ \delta_0(d) \\ \delta_0(e) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

$$\nabla^{11}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\\ \delta_1(e)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -2y^3 & -3x^2\\ 0 & 0 & -y^3 & 0 & 0\\ 0 & 0 & 0 & 0 & -3y^3\\ 0 & 0 & y^2 & 0 & 0\\ 0 & y & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$$

$$\nabla^{12}_{\delta_0}\begin{pmatrix} a\\ b\\ c\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix}$$

$$\nabla^{12}_{\delta_1}\begin{pmatrix} a\\ b\\ c\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\end{pmatrix} + \begin{pmatrix} 0 & 2x & -2y^3\\ 0 & 0 & 2xy^2\\ 0 & 2y & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix}$$

$$\nabla^{13}_{\delta_0}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\\ \delta_0(d)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$\nabla^{13}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\end{pmatrix} + \begin{pmatrix} 0 & 2x & -2y^3 & 0\\ 0 & 0 & 2xy^2 & -x^2\\ 0 & 2y & 0 & y^2\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$\nabla^{14}_{\delta_0}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\\ \delta_0(d)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$\nabla^{14}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\end{pmatrix} + \begin{pmatrix} 0 & 2x & -2y^3 & 0\\ 0 & 0 & 2xy^2 & 4x^2y\\ 0 & 2y & 0 & -4y^3\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$\nabla^{15}_{\delta_0}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\\ \delta_0(d)\\ \delta_0(e)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 2\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$$

$$\nabla^{15}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\\ \delta_1(e)\end{pmatrix} + \begin{pmatrix} 0 & y^2 & 2x & -xy & -2y^3\\ 0 & 0 & 0 & -y^3 & 0\\ 0 & 0 & 0 & 0 & 2xy^2\\ 0 & -x & 0 & 0 & 0\\ 0 & 0 & 2y & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$$

$$\nabla^{16}_{\delta_0}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\\ \delta_0(d)\\ \delta_0(e)\\ \delta_0(f)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$$

$$\nabla^{16}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\\ \delta_1(e)\\ \delta_1(f)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 2x & -xy & -2y^3 & -3x^2\\ 0 & 0 & 0 & -y^3 & 0 & 0\\ 0 & 0 & 0 & 0 & 2xy^2 & 0\\ 0 & 0 & 0 & 0 & 0 & -3y^3\\ 0 & 0 & 2y & 0 & 0 & 0\\ 0 & y & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$$

$$\nabla^{17}_{\delta_0}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix}\delta_0(a)\\ \delta_0(b)\\ \delta_0(c)\\ \delta_0(d)\\ \delta_0(e)\\ \delta_0(f)\end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&2&0\\ 0&0&0&0&0&3\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$$

$$\nabla^{17}_{\delta_1}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix}\delta_1(a)\\ \delta_1(b)\\ \delta_1(c)\\ \delta_1(d)\\ \delta_1(e)\\ \delta_1(f)\end{pmatrix} + \begin{pmatrix} 0&0&y^2&-xy&0&-3x^2\\ 0&0&0&0&0&-3y^3\\ 0&0&0&-y^3&-2x^2&0\\ 0&0&-x&0&0&-3y^3\\ 0&y^2&0&0&0&0\\ 0&0&0&0&2y^2&0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$$

Explicit calculations show that all these connections are regular.

8 Conclusions

To summarize, these calculations prove the following result:

Theorem 8.1 *Let A be a simple, irreducible curve singularity and M be an IMCM A-module. Then, there exists a regular $Der_k(A)$-connection on M. Furthermore, there exists a regular connection on M if and only if M is free.*

Proof: The first part is obvious. For the last part, assume that there exists a regular connection on M. Then M is locally free, and since A is a local domain of dimension 1, M is locally free if and only if M is a free A-module. $\square$

Let us now recapture our first example, where $B = k[x,y]/(x^3 + y^4)$ and N is a B-module given by its free resolution. Let $m = (x,y)$ be the maximal ideal corresponding to the singular point of $Spec(B)$, and let $A = \hat{B}$ be the completion of B in the m-adic topology. Then notice that A is the simple singularity E_6 and that $M = \hat{N} = N \otimes_B A$ is the IMCM A-module M_6.

Assume that $k = \mathbf{C}$, the complex numbers, for the rest of this section. In this case, we want to associate a monodromy operator to every regular $Der_k(A)$-connection on M. For the class of IMCM A-modules M over simple curve singularities A, this is not possible. Instead, we want to consider another class of modules arising from this class in a very natural way, motivated from our example:

To each simple, irreducible curve singularity $A = k[[x,y]]/(f)$, we might associate the corresponding f from the above mentioned list. Notice that f is a polynomial in $k[x,y]$. To each simple, irreducible curve singularity A, we therefore associate the finitely generated k-algebra $B = k[x,y]/(f)$. Furthermore, every IMCM A-module M has a minimal free resolution induced by one matrix factorization of f over $k[[x,y]]$ mentioned above. But all these matrix factorizations are also matrix factorizations of f over $k[x,y]$. Since $(f)/(f^2)$ is B-free, every matrix factorization of f over $k[x,y]$ induces a free resolution of some finitely generated B-module N.

In this way, we get a finitely generated k-algebra B for each simple curve singularity A, and a finitely generated B-module N for each IMCM A-module M. From now on, we will consider the class of k-algebras B and B-modules N obtained in this manner. Notice that this new class of modules has the nice property that every module has a free resolution induced by a matrix factorization.

Let B be a k-algebra and N be a B-module of the type mentioned above. Let A be the corresponding simple curve singularity, and M be the corresponding IMCM A-module. Then, there is a 1-1 correspondence between $Der_k(B)$-connections on

N and $Der_k(A)$-connections on M, and this correspondence preserves regularity. Hence, we have a regular $Der_k(B)$-connection on N, of the exact same form as the corresponding $Der_k(A)$-connection on M mentioned above.

Notice that for every pair (B, N) of the type mentioned, $B_{\{x\}} \cong k[t, t^{-1}]$. Furthermore, there exist isomorphisms $\phi : N_{\{x\}} \to k[t, t^{-1}]^n$ for $n = rk(N)$. When we have fixed one such isomorphism, every regular $Der_k(B)$-connection on N gives rise to a regular $Der_k(B_{\{x\}})$-connection on $k[t, t^{-1}]^n$, or equivalently an action on $k[t, t^{-1}]^n$ with derivation property with respect to $\frac{d}{dt}$. This corresponds to n linear, first order differential equations on $\mathbf{C}^*$. It is easy to see that in each case, these equations has n linearly independent analytic solutions in a neighbourhood of each point in $\mathbf{C}^*$, when the connection ∇ is regular. Hence, we may associate a monodromy operator to every regular $Der_k(B)$-connection on N.

Explicit computations show that if (B, N) is a pair of the type described, and ∇ is the regular $Der_k(B)$-connection on N mentioned earlier, the monodromy operator associated to ∇ is trivial. Consequently, we have proven the following result:

Theorem 8.2 *Let B be a finitely generated k-algebra, and N be a finitely generated B-module of the type mentioned above. Then, there exists a regular $Der_k(B)$-connection on N, which has trivial monodromy.*

Acknowledgements

I wish to thank my supervisor, professor O.A. Laudal for introducing me to the subject of D-modules during my graduate studies at the Department of Mathematics, University of Oslo. This paper is a continuation of the work I did as a graduate student, and parts of it are based on Laudals ideas and works. In particular, the obstruction theory for connections on modules is almost completely due to him.

References

[1] V.I. Arnol'd. *Critical points of smooth functions.* Proc. Int. Cong. Math. Vancouver 1974, Vol. 1, p. 19-39.

[2] M.F. Atiyah, I.G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley 1969.

[3] J.E. Bjørk. *Rings of differential operators.* North-Holland 1979.

[4] R.-O. Buchweitz, G.-M. Greuel, F.-O. Schreyer. *Cohen-Macaulay modules on hypersurface singularities II.* Invent. Math. 88 (1987), p. 165-182.

[5] Marc Chamarie, Ian M. Musson. *Rings of differential operators on one dimensional algebras.* Pac. J. of Math. 147 (1991), Vol. 2, p. 269-290.

[6] Pierre Deligne. *Equations Différentielles à Points Singuliers Réguliers.* Springer 1970.

[7] David Eisenbud. *Homological algebra on a complete intersection, with an application to group representations.* Trans. Am. Math. Soc. 260 (1980), p. 35-64.

[8] G.-M. Greuel, H. Knörrer. *Einfache Kurvensingularitäten und torsionsfreien Moduln.* Math. Ann. 270 (1985), p. 417-425.

[9] R. Hartshorne. *Algebraic Geometry.* Springer 1977.

[10] Horst Knörrer. *Cohen-Macaulay modules on hypersurface singularities I*. Invent. Math. 88 (1987), p. 153-164.

[11] O.A. Laudal. *A framework for change*. In preparation.

[12] Hideyuki Matsumura. *Commutative ring theory*. Cambridge 1986.

[13] J.C. McConnell, J.C. Robson. *Noncommutative Noetherian rings*. Wiley 1987.

[14] K.R. Mount, O.E. Villamayor. *On a conjecture of Y. Nakai*. Osaka J. Math. 10 (1973), p. 325-327.

[15] Ian M. Musson. *Rings of differential operators and zero divisors*. J. of Algebra 125 (1989), p. 489-501.

[16] Yuji Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings*. Cambridge 1990.