LECTURE 1

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MATHEMATICS

PLAN:

1. Matrices and matrix algebra
   - Linear systems - Gaussian elimination - rank
2. Eigenvalues and eigenvectors
   - Diagonalization
3. Definiteness of symmetric matrices
   - Quadratic forms

Reading:

[FMEA] 1.1-1.7
[ME] 6-9,23
([S] 1.3, 1.5)

① Matrices and matrix algebra

An $m \times n$-matrix A is a rectangular array (with m rows and n columns) of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

A (column) vector is an $m \times 1$-matrix

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Operations on matrices:

* Addition / subtraction: $\begin{cases} A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \\ A - B = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}) \end{cases}$

  - defined when A, B has same size
  - computed position by position

* Scalar multiplication: $r \cdot A = r \cdot (a_{ij}) = (r a_{ij})$
  - defined when r is scalar (number)
  - computed position by position

* Multiplication: $\underset{(m \times n)}{A} \cdot \underset{(n \times p)}{B} = (a_{ij}) \cdot (b_{ij}) = (c_{ij})$, where

  $$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

  - defined when $\#\text{cols}(A) = \#\text{rows}(B)$
  - not commutative: $AB \neq BA$

* Transpose: $\underset{(m \times n)}{A} \rightsquigarrow A^T = (a_{ij})^T = \underset{(n \times m)}{(a_{ji})}$

Special matrices:

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$ zero matrix (any size)

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$ identity matrix $(n \times n)$

Property: $\boxed{A \cdot I = A \quad I \cdot A = A}$

Square matrix: #rows = #cols

Diagonal: $A = (a_{ij})$ square such that $a_{ij} = 0$ for $i \neq j$ $\qquad A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$

Symmetric: $A = (a_{ij})$ square such that $A^T = A$

Upper/lower triangular: $\begin{pmatrix} d_1 & & * \\ & d_2 & \\ 0 & \ddots & d_n \end{pmatrix}$, $\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ * & & d_n \end{pmatrix}$

Inverse matrix: A $n \times n$-matrix

An inverse of A is a matrix B such that $A \cdot B = B \cdot A = I_n$

If it exists, it is unique and written A^{-1} (instead of B)

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\Longrightarrow$ $ad - bc = 0$: A^{-1} does not exist

2x2-matrix $\qquad ad - bc \neq 0$: $\boxed{A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Determinant: A $n \times n$-matrix $\rightsquigarrow$ $\det(A) = |A|$ is a number

Determinant $\det(A)$ defined inductively:

i) $n = 1$: $A = (a) \to |A| = a$

ii) General case: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \Longrightarrow |A| = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} & \cdots \\ a_{32} & \ddots & \\ \vdots & & \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} & \cdots \\ a_{31} & a_{33} & \cdots \\ \vdots & & \ddots \end{vmatrix} + \cdots$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \boxed{\det(A) = |A| = a \cdot d - bc}$

2x2

Property: A $n \times n$-matrix

A^{-1} exists $\Longleftrightarrow |A| \neq 0$

Linear systems and Gaussian elimination

A linear system is a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Matrix form:

$$A \cdot \underline{x} = \underline{b}$$

Augmented matrix:

$$(A \,|\, \underline{b})$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Gaussian elimination is a solution method; see [LSGE] for details.

Example:

$$\begin{cases} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{cases} \Rightarrow$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} -1 \\ -1 \\ \end{array}$$

pivot positions

elementary row operations

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ -2 \\ \end{array}$$

$$\downarrow$$

Echelon form (zeros under all pivots)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\Downarrow$$

Linear system:

$$\begin{aligned} x + y + z &= 3 \quad \leftarrow x = 1 \\ y + 3z &= 4 \quad \leftarrow y = 1 \\ 2z &= 2 \quad \leftarrow z = 1 \end{aligned}$$

Solution: $\underline{x = 1, y = 1, z = 1}$

Rank

The <u>rank</u> of a matrix A is the number of pivot positions in its echelon form. It is written $\text{rk } A$.

Let $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ be n m-vectors. The vectors are called <u>linearly independent</u> if $x_1 \underline{v_1} + x_2 \underline{v_2} + \cdots + x_n \underline{v_n} = \underline{0}$ has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$, and <u>linearly dependent</u> if there are non-trivial solutions.

<u>Fact</u>: When $A = (\underline{v_1} \,\vdots\, \underline{v_2} \,\vdots\, \cdots \,\vdots\, \underline{v_n})$, then $\text{rk } A$ is the maximal number of linearly independent vectors among $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$.

Let A be an $m \times n$-matrix. A $k \times k$-submatrix of A is a matrix B obtained by selecting k rows $(i_1, i_2, \ldots, i_k)$ and k columns $(j_1, j_2, \ldots, j_k)$ from A. <u>A minor of order k from A</u> is the determinant of a $k \times k$-submatrix.

<u>Fact</u>: $\text{rk } A$ is the maximal order k such that there exists a non-zero minor of order k;

$$\text{rk } A = \max \{ k : M_k \neq 0 \text{ for a minor } M_k \text{ of order } k \}$$

<u>Example</u>:

$$A = \begin{pmatrix} 1 & 7 & 5 \\ 3 & 4 & -2 \end{pmatrix} \qquad \begin{vmatrix} 1 & 7 \\ 3 & 4 \end{vmatrix} = 4 - 21 = -17 \neq 0$$

$$\Rightarrow \text{rk } A = \underline{2}$$

A <u>principal minor</u> is a minor obtained by selecting k rows $i_1, i_2, \ldots, i_k$ and the same columns $j_1 = i_1, j_2 = i_2, \ldots, j_k = i_k$.

A <u>leading principal minor</u> is a minor obtained by selecting rows $1, 2, 3, \ldots, k$ and columns $1, 2, \ldots, k$.

(2) Eigenvalues and eigenvectors

Let A be an $n \times n$-matrix.

A number λ is an eigenvalue for A if $A \cdot \underline{v} = \lambda \cdot \underline{v}$ has a non-trivial solution $\underline{v} \neq \underline{0}$. In that case, the eigenspace of eigenvectors of A with eigenvalue λ is

$$E_\lambda = \{ \underline{v} : A\underline{v} = \lambda \underline{v} \}$$

The equation $A\underline{v} = \lambda \underline{v}$ can be rewritten as $(A - \lambda I)\underline{v} = \underline{0}$.

Fact: The eigenvalues of A are the solutions of the characteristic equation $\det(A - \lambda I) = 0$. It is a polynomial equation of order n.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \det(A-\lambda I) = \underbrace{\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}}_{A-\lambda I} = (1-\lambda)\cdot(1-\lambda) - 2^2 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\underline{\lambda = 3}, \ \underline{\lambda = -1}$$

Fact: The characteristic equation is $(-1)^n \cdot \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1}\lambda + c_n = 0$ where $c_1 = tr(A)$ and $c_n = \det(A)$

$(tr\, A = a_{11} + a_{22} + \dots + a_{nn})$

Fact: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the n eigenvalues of A, then

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$$
$$tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$\det(A - \lambda I) =$

$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

Fact: If A is symmetric, then A has n real eigenvalues (counted with multiplicity).

If A is symmetric, then we have

$$\det(A-\lambda I) = (\lambda-\lambda_1)^{n_1}\cdot(\lambda-\lambda_2)^{n_2}\cdot\ldots\cdot(\lambda-\lambda_k)^{n_k}$$

with $n_1+n_2+\ldots+n_k=n$. The number n_i is the (algebraic) multiplicity of λ_i.

Fact:

If λ is an eigenvalue of A of multiplicity m, then the equation $(A-\lambda I)\cdot \underline{v}=\underline{0}$ has at most m degrees of freedom.

Example:

$$A=\begin{pmatrix}3&1\\0&3\end{pmatrix}$$

$$A=\begin{pmatrix}2&3\\3&2\end{pmatrix}$$

A square matrix A is called diagonalizable if there is diagonal matrix D and a invertible P such that $A=P\cdot D\cdot P^{-1}$.

Fact: If A has eigenvalues $\lambda_1,\ldots,\lambda_n$ and linearly independent eigenvector $\underline{v}_1,\ldots,\underline{v}_n$ then $A=PDP^{-1}$ if

$$D=\begin{pmatrix}\lambda_1&&&0\\&\lambda_2&&\\&&\ddots&\\0&&&\lambda_n\end{pmatrix},\quad P=\left(\underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \ldots \,\vdots\, \underline{v}_n\right)$$

If $\lambda_1,\lambda_2,\ldots,\lambda_k$ are k distinct eigenvalues for A, with multiplicities $m_1,m_2,\ldots,m_k$, then A is diagonalizable if and only if the following conditions hold:

i) $m_1+m_2+\ldots+m_k=n$

ii) The equation $(A-\lambda_i I)\underline{v}=\underline{0}$ has m_i degrees of freedom for $i=1,2,\ldots,k$.

Example:

$A = \begin{pmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{pmatrix}$ We want to compute A^n when $n \gg 0$

If A diagonalizable, then $A = PDP^{-1}$, and we have

P: inv. matrix, D: diagonal matrix

$$A^n = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$$
$$= PD^nP^{-1}$$
$$= P \cdot \begin{pmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n & \\ 0 & & \ddots \; \lambda_k^n \end{pmatrix} P^{-1} \quad \text{if } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_k \end{pmatrix}$$

Eigenvalues: $\lambda = 1$, $\lambda = 0.75$

$\underline{v}_1$ $\quad$ $\underline{v}_2$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}$

$P = (\underline{v}_1 \mid \underline{v}_2)$

$$A^n = P \cdot \begin{pmatrix} 1^n & 0 \\ 0 & 0.75^n \end{pmatrix} \cdot P^{-1} \xrightarrow[n \to \infty]{} P \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot P^{-1}$$

(3) Definiteness

A quadratic form in n variables is a polynomial function where all terms have degree two,

$$Q(x_1, x_2, \ldots, x_n) = a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{nn}x_n^2$$

$$\parallel$$

$$Q(\underline{x})$$

A quadratic form can be written in matrix form

$$Q(\underline{x}) = (x_1\ x_2 \ldots x_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{x}^T A \underline{x}$$

for a unique symmetric $n \times n$-matrix A.

Definition: Q and A are called

positive semidefinite	$\Leftrightarrow$ $\underline{x}^T A \underline{x} \geq 0$	for all $\underline{x}$
negative –"–	$\underline{x}^T A \underline{x} \leq 0$	–"–
indefinite	$\underline{x}^T A \underline{x}$	have positive and negative values
positive definite	$\Leftrightarrow$ $\underline{x}^T A \underline{x} > 0$	for all $\underline{x} \neq \underline{0}$
negative –"–	$\underline{x}^T A \underline{x} < 0$	–"–

Fact: If A is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, then we have

$\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ $\Leftrightarrow$	A positive semidefinite
> 0	positive definite
≤ 0	negative semidefinite
< 0	negative definite
$\lambda_i > 0,\ \lambda_j < 0$	indefinite

Effective method:

Let A be symmetric $n\times n$-matrix, let $D_1, D_2, \ldots, D_n$ be its leading principal minors, and let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be any of its principal minors.

A positive definite $\iff D_1, D_2, \ldots, D_n > 0$

A negative " $\quad D_1 < 0, D_2 > 0, D_3 < 0, \ldots$ (that is, $(-1)^i D_i > 0$)

A positive semidefinite $\iff \Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$ for all principal minors

A negative " $\quad \Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$ — " —