Lecture 8

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DKE 7007

Mathematics

Plan:

① Optimal control theory - continuous case

- Pontryagin's maximum principle
- Sufficient conditions

~~Remember: Thu and Fri 14-16 at C2-095~~

Reading

[FMEA] 9 (8.10)

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①

Problem:

$$\boxed{\max \int_{t_0}^{t_1} f(t,x,u)\,dt \quad \text{such that} \begin{cases} x(t_0)=x_0 \\ \dot{x}=g(t,x,u) \\ u \in U \subseteq \mathbb{R} \\ \text{Either: } \begin{array}{l} a)\ x(t_1)=x_1 \\ b)\ x(t_1) \text{ is free} \end{array} \end{cases}}$$

$u = u(t)$ control variable $\quad u: [t_0,t_1] \to U \quad$ ($U \subseteq \mathbb{R}$ control region)

$x = x(t)$ state variable $\quad x: [t_0,t_1] \to \mathbb{R}$

When u is chosen, then x is given by the ODE $\dot{x} = g(t,x,u)$ and initial condition $x(t_0)=x_0$. We may or may not have an additional condition on $x(t_1)$.

Hence, for each u, we may compute $\int_{t_0}^{t_1} f(t,x,u)\,dt$, and we want to find u such that this is maximal.

Necessary conditions:

The Hamiltonian $H(t,x,u,p) = p_0 f(t,x,u) + p \cdot g(t,x,u)$

where $p_0 \in \mathbb{R}$ ~~and~~ $p: [t_0,t_1] \to \mathbb{R}$ is a function in t.

Maximum principle: (Pontryagin)

If x^*, u^* is an optimal pair, then there is a function $p(t)$ and a number p_0 such that $(p_0, p(t)) \neq (0,0)$ for all $t \in [t_0,t_1]$, such that

(A) $u \mapsto H(t,x^*,u,p)$ has a maximum at $u = u^*$

(B) $\dot{p}(t) = -H'_x(t,x^*,u^*,p)$

(C) Transversality: $\begin{cases} \text{a) } \underline{x(t_1)=x_1}: & \text{no condition} \\ \text{b) } \underline{x(t_1)=\text{free}}: & p(t_1)=0 \end{cases}$

* We may assume $p_0=0$ or $p_0=1$. If $x(t_1)$ is free, then we may assume $p_0=1$.

Sufficient conditions:

Theorem: (Mangasarian)
Suppose that (x^*,u^*) is admissible and satisfies (A)-(C) above with $p_0=1$.
If U is convex and $(x,u) \mapsto H(t,x,u,p)$ is concave for all $t \in [t_0,t_1]$, then (x^*,u^*) is an optimal pair.

Ex I: $\max \int_0^T (1 - tx - u^2)\,dt$, $\quad \dot{x} = u$, $x(0) = x_0$, $U = \mathbb{R}$ $\quad (x_0, T \text{ given})$

Necessary condition:

$H = p_0 \cdot (1 - tx - u^2) + p \cdot u$

$p_0 = 0$: $p(t) \neq 0$

(B) $\dot{p} = 0 \Rightarrow p(t) = c \neq 0$ constant

$H = pu = c \cdot u$ has no max as a fn. in u, so (A) not satisfied

$\Downarrow$

no solution with $p_0 = 0$

$p_0 = 1$: $H = 1 - tx - u^2 + pu$

$\dot{x} = u$

$x(0) = x_0$

(A) $\frac{\partial H}{\partial u} = p - 2u = 0 \Rightarrow u = \frac{1}{2}p$

(B) $p' = -(-t) = t \Rightarrow p = \frac{1}{2}t^2 + C$

(C) $p(T) = 0 \Rightarrow C = -\frac{1}{2}T^2 \Rightarrow p(t) = \frac{1}{2}t^2 - \frac{T^2}{2} \Rightarrow u = \frac{1}{4}t^2 - \frac{T^2}{4}$

$\dot{x} = u = \frac{1}{4}t^2 - \frac{T^2}{4} = \frac{1}{12}t^3 - \frac{T^2}{4}t + C$

$x(0) = x_0 \Rightarrow C = x_0 \Rightarrow x = \frac{1}{12}t^3 - \frac{T^2}{4}t + x_0$

$U = \mathbb{R}$ convex & $H = 1 - tx - u^2 + pu$ concave in (x, u) $\Rightarrow$ $u = \frac{1}{4}t^2 - T^2/4$, $x = \frac{1}{12}t^3 - \frac{T^2}{4}t + X$ is the maximizer