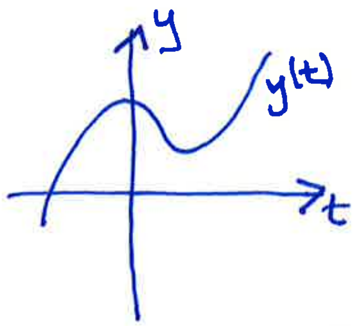
Plan:

1. Differential equations
2. Systems of differential equations
3. Linearizations

1 Differential equations (ordinary)

An ordinary differential equation (ODE) is an an equation in $t, y, y', y'', \ldots$

Unknown function: $y = y(t)$



Ex: $y \cdot y' = t \iff y(t) \cdot y'(t) = t$

$$y' = \frac{t}{y}$$

The order of an ODE is the highest order derivative that appears in the equation.

First order: $y' = F(y, t)$

Cases:/ solution methods:
- separation: $y' = f(y) \cdot g(t)$
- integrating factor: $y' + a(t) y = b(t)$
- linear methods

Defn: A first order ODE is autonomous if $\boxed{y' = F(y)}$

Linear first order autonomous ODE:

$y' = F(y) = ay + b$

$\boxed{y' = ay + b}$ where a, b are constants $(a \neq 0)$

Steady state = equilibrium state:

$F(y) = 0$: $ay + b = 0$

$y_e = -\frac{b}{a}$ ← constant

Change of variables: $\boxed{z = y - y_e}$ $z' = y'$

$y' = ay + b \rightsquigarrow z' = a \cdot (z + y_e) + b$

$= az + \underbrace{ay_e + b}_{0}$

$z' = az$ $z_e = 0$

$z' = az$

$z' - az = 0$

Char. eqn: $r - a = 0$

$r = a$

$\rightsquigarrow \underline{z = C \cdot e^{+at}}$

$\Downarrow$

$y = z + y_e$

$\boxed{y = C \cdot e^{(+at)} + y_e}$

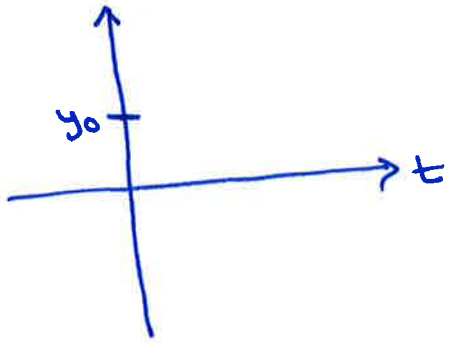
$$y' = ay + b \quad \Longrightarrow \quad y = Ce^{+at} - \frac{b}{a}$$

general solution

$$y(0) = y_0$$

$$y_0 = C \cdot e^{+a\cdot 0} - \frac{b}{a}$$

$$y_0 = C - \frac{b}{a} \Rightarrow C = y_0 + \frac{b}{a}$$



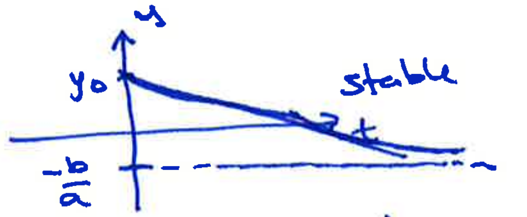
Particular solution:

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{+at} - \frac{b}{a}$$

Stable steady state:

If y_0 is close to (but not equal to) y_e, the solution $y(t) \to y_e$ as $t \to \infty$

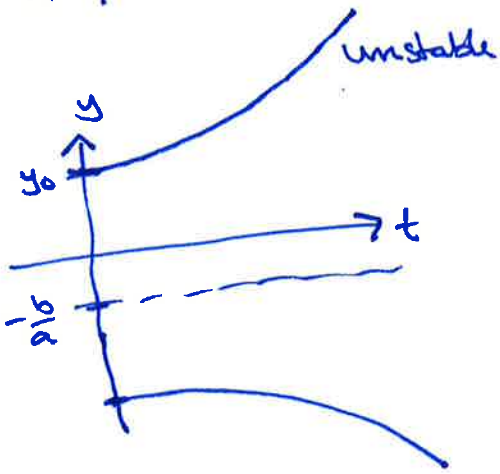
$a<0$:



Unstable steady state

if y_0 is close to (but not equal to) y_e, the solution $y(t)$ would move away from y_e

$a>0$:



② System of linear differential equations:

$$y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2$$
$$y_3' = \quad \vdots \qquad \qquad \vdots$$
$$\vdots$$
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + b_n$$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}: \qquad \underline{y}' = A\cdot\underline{y} + \underline{b}$$

Ex: $y_1' = y_1 - 3y_2 + 2$ $\qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$y_2' = 2y_1 - 4y_2 + 2$

Steady state: $\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff A\cdot\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \underline{b} = \underline{0}$

$$A\cdot\underline{y} = -\underline{b}$$

Ex: $y_1 - 3y_2 + 2 = 0$ $\qquad \left(\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -4 & -2 \end{array}\right)] \, -2$

$2y_1 - 4y_2 + 2 = 0$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 2 & 2 \end{array}\right) \qquad y_1 = 1,\; y_2 = 1$$

$$\underline{y}_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Change of variables:

$\underline{y}' = A\cdot\underline{y} + \underline{b}$

$\underline{z} = \underline{y} - \underline{y_e}$

$\Updownarrow$

$z_1 = y_1 - (y_e)_1$
$z_2 = y_2 - (y_e)_2$
$\vdots$
$z_n = y_n - (y_e)_n$

$$\underline{z}' = \underline{y}' = A\cdot\underline{y} + \underline{b}$$
$$= A\cdot(\underline{z} + \underline{y_e}) + \underline{b}$$
$$= A\underline{z} + (A\cdot\underline{y_e} + \underline{b})$$
$$= A\underline{z}$$

$\underline{z} = \underline{y} - \underline{y_e}$:

$\underline{y}' = A\underline{y} + \underline{b} \rightsquigarrow \underline{z}' = A\underline{z}$

Ex: $\underline{y}' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad \underline{y_e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{z} = \underline{y} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\quad \boxed{\underline{z}' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}\cdot\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}$

Consider $\underline{z}' = A\underline{z}$ and assume that A is diagonalizable with $P^{-1}AP = D$.

New variables: $P\underline{u} = \underline{z} \iff \underline{u} = P^{-1}\cdot\underline{z}$

$$P\underline{u} = \underline{z} \qquad (P\underline{u})' = \underline{z}' = A\underline{z}$$

$$P\cdot\underline{u}' = A\underline{z} = A\cdot P\underline{u}$$

$$\underline{u}' = P^{-1}AP\underline{u} = D\underline{u}$$

$$\boxed{\underline{u}' = D\cdot\underline{u}} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\begin{aligned} u_1' &= \lambda_1 \cdot u_1 & \Rightarrow\ u_1 &= c_1\cdot e^{\lambda_1 t} \\ u_2' &= \lambda_2 u_2 & u_2 &= c_2 \cdot e^{\lambda_2 t} \\ &\vdots & &\vdots \\ u_n' &= \lambda_n u_n & u_n &= c_n\cdot e^{\lambda_n t} \end{aligned} \qquad P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$$

$$\underline{u} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \Rightarrow \underline{z} = P\cdot\underline{u} = P\cdot\begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= c_1 e^{\lambda_1 t}\cdot \underline{v_1} + c_2 e^{\lambda_2 t}\cdot\underline{v_2} + \dots + c_n e^{\lambda_n t}\cdot \underline{v_n}$$

$\underline{y} = \underline{z} + \underline{y_e}$:

$$\boxed{\underline{y} = c_1\cdot\underline{v_1}\cdot e^{\lambda_1 t} + \dots + c_n \underline{v_n} e^{\lambda_n t} + \underline{y_e}}$$

How to solve $\underline{z}' = A\underline{z}$:

A diagonal:
$$\begin{pmatrix} z_1' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$z_1' = a_1 \cdot z_1$$
$$z_2' = a_2 z_2$$
$$\vdots$$
$$z_n' = a_n z_n$$

Try to diagonalize A:

Ex: $A = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix}$

$|A - \lambda I| = 0$ $\quad \begin{vmatrix} 1-\lambda & -3 \\ 2 & -4-\lambda \end{vmatrix} = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$
$$\lambda = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} = -2, -1$$
$$\underline{\lambda = -2}, \quad \underline{\lambda = -1}$$

$D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ $\quad \boxed{P^{-1}AP = D}$

$E_{-2} = \text{span}(\underline{v}_1)$

E_{-2}: $\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \leftarrow A - \lambda I \rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_{-1}: $\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \leftarrow A - \lambda I$ $\quad \underline{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$E_{-1} = \text{span}(\underline{v}_2)$

Ex: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad y_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $\quad \underline{z} = y - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$

$\Downarrow$

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{-2t} + c_2 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot e^{-t}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{\underline{c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

$y_1 = c_1 e^{-2t} + 3c_2 e^{-t} + 1$

$y_2 = c_1 e^{-2t} + 2c_2 e^{-t} + 1$

Note: Any n'th order linear differential eqn. of the form

$y^{(n)} + c_1 y^{(n-1)} + \dots + c_{n-1} y' + c_n y = d$

can be written as an $n \times n$-system of linear diff.-eqn.

Ex: $y'' - 3y' + 2y = 5 \Rightarrow$ $z = y'$ $\quad \begin{aligned} y' &= z \\ z' &= 3y' - 2y + 5 = \underline{-2y + 3z + 5} \end{aligned}$

$\underline{\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix}}$

Global asymptotically stable if $y \to \bar{y}$ ye when $t \to \infty$ for all initial states y_0.

$$\Updownarrow$$

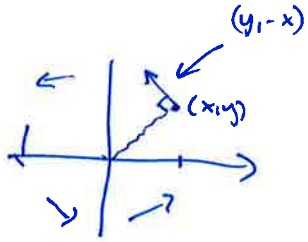
$\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_n < 0$

If λ_i are complex eigenvalues, the condition becomes:
the real part of λ_i is negative for all i

In the case $n=2$:

$$\lambda_1, \lambda_2 < 0 \iff \begin{cases} \underbrace{\operatorname{tr} A}_{=\lambda_1+\lambda_2} < 0 \\ \underbrace{\det A}_{=\lambda_1 \cdot \lambda_2} > 0 \end{cases}$$

Ex: $y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$



$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\lambda^2 + 1 = 0$

$\lambda^2 = \pm\sqrt{-1} = \pm i$

$\underline{\lambda_1 = i}, \underline{\lambda_2 = -i}$

(complex eigenvalues)

This characterization also holds for complex eigenvalues

Complex numbers: $z = a + ib$, where $a, b \in \mathbb{R}$, "$i = \sqrt{-1}$" (ie $i^2 = -1$)

a: real part, b: imaginary part

$z^2 - 2z + 5 = 0$:

$$z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2}$$

$$= 1 \pm \frac{\sqrt{16} \cdot i}{2} = 1 \pm 2i$$

$z_1 = \underline{1+2i}$, $z_2 = \underline{1-2i}$

3 Linear approximations

$$\left.\begin{aligned} y_1' &= F(y_1, y_2) \\ y_2' &= G(y_1, y_2) \end{aligned}\right\}$$ where F, G are general (non-linear) functions

Steady state: $y = \bar{y}_{\to y_e}$ s.t. $F(\bar{y}) = G(\bar{y}) = 0$.

Linearization:

$$y_1' = F'_{y_1}(\bar{y}) \cdot (y_1 - \bar{y}_1) + F'_{y_2}(\bar{y}) \cdot (y_2 - \bar{y}_2)$$
$$y_2' = G'_{y_1}(\bar{y})(y_1 - \bar{y}_1) + G'_{y_2}(\bar{y}) \cdot (y_2 - \bar{y}_2)$$

$\underline{y}' = A \cdot (\underline{y} - \bar{y})$ or $\boxed{\underline{z}' = A \cdot \underline{z}}$ with $\underline{z} = \underline{y} - \bar{y}$ and

$$A = \begin{pmatrix} F'_{y_1} & F'_{y_2} \\ G'_{y_1} & G'_{y_2} \end{pmatrix}$$

Ex:
$$x' = x - 3y + 2x^2 + y^2 - xy$$
$$y' = 2x - y - e^{x+y} + 1$$

$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is one steady state (there may be others)

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Linearization:

$$\underline{z}' = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \underline{z}$$

$$\left.\begin{aligned} \det A &= -2 + 3 = 1 > 0 \\ \operatorname{tr} A &= 1 + (-2) = -1 < 0 \end{aligned}\right\}$$ Globally asymptotically stable at $(0,0)$.