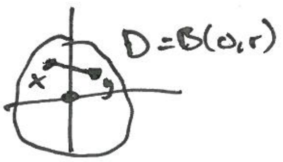
Problem Sheet 4
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Sheet 4

1. Let $D = B(0,r) \subseteq \mathbb{R}^n$. If D is convex, then $B(p,r) = B(0,r) + \{p\}$ is convex, since it is the sum of convex sets.



If $x,y \in D$, then $\lambda x + (1-\lambda)y \in D$ for $\lambda \in [0,1]$:

$$\|x\| < r,\ \|y\| < r \Rightarrow \|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\|$$
$$= \lambda \cdot \|x\| + (1-\lambda)\|y\| < \lambda r + (1-\lambda) r$$
$$= r.$$

This means that $D = B(0,r)$ is convex.

2. a) $f(x,y) = e^{xy} - 1$

$$D^2 f(x,y) = \begin{pmatrix} y^2 e^{xy} & (1-xy)e^{xy} \\ (1-xy)e^{xy} & x^2 e^{xy} \end{pmatrix}$$

$D_1 = y^2 e^{xy} \geq 0$, $\Delta_1 = x^2 e^{xy} \geq 0$

$D_2 = (-1-2xy)e^{2xy}$

Since $D_2 < 0$ for some (x,y), f is <u>not convex, not concave</u>

b) $f(x,y,z) = xyz$

$$D^2 f(x,y,z) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

$D_1 = 0$, $\Delta_1 = 0, 0$

$D_2 = -z^2$

$\vdots$

Since $D_2 < 0$ for some (x,y,z), f is <u>not convex, not concave</u>

c) $f(x,y,z) = \dfrac{1}{xyz} = (xyz)^{-1}$

$$D^2 f(x,y,z) = \begin{pmatrix} -\dfrac{2y^2z^2}{x^3y^3z^3} & \cdots \\ \vdots & \end{pmatrix} = \begin{pmatrix} +\dfrac{2}{x^3yz} & \cdots \\ \vdots & \end{pmatrix}$$

$D_1 = \dfrac{2}{x^3yz}$

Since $D_1 > 0$ for some (x,y,z) and positive for some (x,y,z),

f is <u>not convex, not concave</u>

3. $D = \{(x,y) : x \geq 0, y \geq 0\} \longrightarrow \mathbb{R}$ for $a,b,C > 0$.

$$(x,y) \longmapsto C x^a y^b$$

$$\left.\begin{array}{l} f'_x = C \cdot a x^{a-1} y^b \\ f'_y = C \cdot b x^a y^{b-1} \end{array}\right\} \quad D^2 f(x,y) = \underline{\begin{pmatrix} a(a-1)x^{a-2}y^b & ab x^{a-1} y^{b-1} \\ ab x^{a-1} y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix} \cdot C}$$

$$D_1 = C a(a-1) x^{a-2} y^b$$

$$D_2 = C^2 x^{2a-2} y^{2b-2} \left(ab(a-1)(b-1) - a^2 b^2 \right)$$

$$= C^2 ab\, x^{2a-2} y^{2b-2} (1-a-b)$$

If $1-a-b<0$, then f is <u>neither convex nor concave</u> since $D_2 < 0$.

If $1-a-b>0$, then $a<1$ and $D_1 < 0, D_2 > 0$, so f is <u>(strictly) concave</u>.

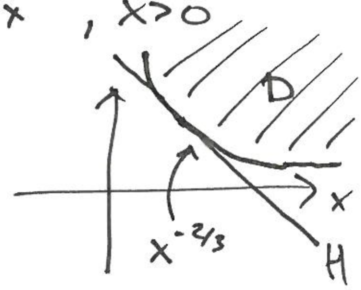
If $1-a-b=0$, then $D_2 = 0$, $D_1 < 0$, $\Delta_1 < 0$, hence f is <u>concave</u>.

f concave: $a+b \leq 1$

f convex: <u>never</u>

4. $D = \{(x,y) : x^2 y^3 \geq 1, x>0, y>0\}$

$\partial D = \{(x,y) : x^2 y^3 = 1, x > 0\}$ = graph of $y = x^{-2/3}$, $x>0$



D is convex $\Leftrightarrow$ $y = x^{-2/3}$ convex function

$y' = -\frac{2}{3} x^{-5/3} \Rightarrow y'' = \frac{10}{9} x^{-8/3} > 0$

hence $y = x^{-2/3}$ is <u>convex</u>.

$$d(x) = \| (x, x^{-2/3}) - (0,0) \| = \sqrt{x^2 + x^{-4/3}} \Rightarrow d^2(x) = x^2 + x^{-4/3}, \; x>0$$

$$(d^2(x))' = 2x - \frac{4}{3} x^{-7/3} = 0 \qquad 2x^{-7/3}\left(x^{10/3} - 1\right) = 0$$

$$2x = \frac{4}{3} x^{-7/3}$$

$$x^{10/3} = 2/3 \Rightarrow x = (2/3)^{3/10} \approx \underline{0.885} \quad \text{global min. on } x>0$$

$y = \left((2/3)^{3/10}, (2/3)^{-1/5} \right)$ is closest to $(0,0)$

$H: \ \underline{p} \cdot \underline{x} = a \quad \text{with} \quad p = \underline{y} - \underline{\otimes} = \left((2/3)^{3/10}, \ (2/3)^{-1/5} \right)$

$a = p \cdot y = (2/3)^{3/5} + (2/3)^{-2/5}$

$= (2/3)^{-2/5} (2/3 + 1) = 5/3 \cdot (2/3)^{-2/5}$

$H: \ \left(\frac{2}{3}\right)^{3/10} \cdot x + \left(\frac{2}{3}\right)^{-\frac{1}{5}} \cdot y = \frac{5}{3} (2/3)^{-2/5}$

$\underline{0.885 x + 1.084 y = 1.960}$