LECTURE 8

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KAROLINE MOE

DRE 7017

OPTIMAL CONTROL THEORY IN CONTINUOUS TIME
STANDARD PROBLEM

FMEA 9 (8,10)

OPTIMAL CONTROL THEORY IN CONTINUOUS TIME
STANDARD PROBLEM

$$(**) \quad \max \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt \qquad \text{when} \quad \begin{array}{l} x(t_0) = x_0 \\ x' = g(t,x,u) \\ u \in U \text{ fixed ctrl-region} \\ \text{a) } x(t_1) = x_1 \\ \text{b) } x(t_1) \text{ free (c) in 9.4} \end{array}$$

Seek optimal pair (x^*, u^*) among all admissible pairs (x,u) that satisfy

METHOD: Hamiltonian

$$H = H(t,x,u,p) = p_0 \cdot f(t,x,u) + p \cdot g(t,x,u)$$

with $p = p(t)$ and $p_0 = 0$ or $p_0 = 1$. (9.4.1. FMEA)

PONTRYAGIN'S MAXIMUM PRINCIPLE (Necessary condition)

If (x^*, u^*) is optimal for $(**)$, then there exists a continuous function $p(t)$ and a number p_0 (0 or 1), s.t. for all $t \in [t_0, t_1]$

- $(p_0, p(t)) \neq (0,0)$ and
- (A) u^* maximizes $H(t, x^*, u, p)$ wrt u.

  (B) $p'(t) = -H'_x(t, x^*, u^*, p)$

  (C) a) $x(t_1) = x_1$ no condition

  b) $x(t_1)$ free $p(t_1) = 0$.

NOTE: (1) Only $p_0 = 0$ is unusual! (Any f would give the same...)
In fact, most problems have $p_0 = 1$, and in the book it is assumed that $p_0 = 1$ without proof
"Normal problem"

MANGASARIAN (sufficient condition) (9.4.2 FMEA)

Assume (x^*, u^*) satisfies Ⓐ-Ⓑ-Ⓒ with $p_0 = 1$.
Suppose U is convex and

$$(x,u) \longmapsto H(t,x,u,p)$$

is concave for every $t \in [t_0, t_1]$.
Then (x^*, u^*) is optimal.

APPROACH

a) Maximize $H(t,x,p,u)$ wrt $u \in U \rightarrow \hat{u}(t,x,p)$

b) $x'(t) = g(t, x(t), \hat{u}(t, x(t), p(t)))$
and
$p'(t) = -H'_x(t, x(t), \hat{u}(t, x(t), p(t)), p(t))$

Two diff. eq. solves $x(t)$ and $p(t)$ General solutions

c) Particular solution (constants determined)
by $x(t_0) = x_0$
& Terminal conditions a) $x(t_1) = x_1$ b) $x(t_1) =$ free
Transversality cond. a) $p(t_1)$ no cond. b) $p(t_1) = 0$

NOTE: The ADJOINT FUNCTION $p(t)$

With $V(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))dt$
the optimal value function, we have

$$p(t_0) = \frac{\partial V(x_0, x_1, t_0, t_1)}{\partial x_0}$$

← marginal change as x_0 varies.

↳ computed using Leibniz' formula p. 160 FMEA.

- In general (requires more than we cover here) $p(t)$ is the first-order approximate change in the value function due to an enforced unit ~~jump~~ increase in $x(t)$.
- Shadow price

EX

$$\max \int_0^T (1 - tx - u^2)\,dt \qquad \begin{array}{l} x' = u \\ x(0) = x_0 \\ U = \mathbb{R}\ \text{, convex} \end{array} \qquad (x_0, T) \text{ given}$$

$$H = p_0(1 - tx - u^2) + p \cdot u$$

$p_0 = 1$:

$H'_x = -t$

$H'_u = -2u + p$

$H''_{xx} = 0$

$H''_{xu} = 0$

$H''_{uu} = -2$

$$\begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix}$$

T.2.3.3, FMEA

Δ_1: $0 \quad -2 \quad \leq 0$

$\Delta_2 = 0 \quad \geq 0$

$\Rightarrow H$ concave

$p_0 = 0$:

$$H = p \cdot u$$

(A) • $\dfrac{\partial H}{\partial u} = p$

(B) • $p'(t) = -H'_x = 0 \quad \Rightarrow p(t) = C \in \mathbb{R}$

$H = C \cdot u$ has no maximizer in U, so no solutions with $p_0 = 0$.

$p_0 = 1$:

$$H = (1 - tx - u^2) + p \cdot u$$

(A) • $\dfrac{\partial H}{\partial u} = -2u + p$

$-2\hat{u} + p = 0$

$\hat{u} = \frac{1}{2} \cdot p$

(B) • $p'(t) = -H'_x = t$

$p(t) = \int t\,dt = \frac{1}{2}t^2 + C$ (General sol of p)

(C) Transversatility $x(T)$ free $\Rightarrow p(T) = 0$

$\frac{1}{2}T^2 + C = 0$

$C = -\frac{1}{2}T^2$

$p(t) = \frac{1}{2}t^2 - \frac{1}{2}T^2$ (particular sol of p)

$u^*(t) = \frac{1}{2}p = \frac{1}{4}t^2 - \frac{1}{4}T^2$

$x'(t) = u = \frac{1}{4}t^2 - \frac{1}{4}T^2$

$x(t) = \frac{1}{12}t^3 - \frac{1}{4}T^2 t + C$ (General sol of x)

$x(0) = x_0 = C$, so $x^*(t) = \frac{1}{12}t^3 - \frac{1}{4}T^2 t + x_0$

Since U convex and H concave, (x^*, u^*) is the maximizer

PROBLEM 8-1

$$\max \int_0^2 (3-x^2-u^2)\,dt \qquad \text{when} \qquad \begin{array}{l} x' = u \\ x(0)=1 \\ x(2)=4 \\ U = \mathbb{R} \end{array}$$

$H = p_0(3-x^2-u^2) + pu$

$\underline{p_0 = 0:}$ Same argument as in earlier example.

$\underline{p_0 = 1:}$ $H = 3-x^2-u^2+pu$

Ⓐ Max H wrt u

$H'_u = -2u + p$

$-2\hat{u} + p = 0$

$\hat{u} = \frac{1}{2}p$

Ⓑ $p'(t) = -H'_x = 2x$

Have now a system of linear differential equations:

$\hat{u}' = \frac{1}{2}p'(t) = \frac{1}{2}\cdot 2x = x$

$x' = \hat{u}$

$$\begin{pmatrix} \hat{u} \\ x \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \cdot \begin{pmatrix} \hat{u} \\ x \end{pmatrix}$$ ← No constant term

Find eigenvalues & eigenvectors A:

$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so $\underline{\lambda_1 = 1}$ $\underline{\lambda_2 = -1}$

$E_{\lambda_1}: \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_{\lambda_2}: \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} \hat{u} \\ x \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

(c) I: $x(0) = 1 = C_1 e^0 - C_2 e^{-0} = C_1 - C_2$

II $x(2) = 4 = C_1 e^2 - C_2 e^{-2}$

$$\left.\begin{array}{l} \\ \end{array}\right\} \begin{pmatrix} 1 & -1 \\ e^2 & -e^{-2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

with unique solution

I $C_1 = 1 + C_2$

II $4 = (1+C_2)e^2 - C_2 e^{-2}$

$= e^2 + C_2(e^2 - e^{-2})$

$$\underline{C_2 = \frac{4-e^2}{e^2-e^{-2}}} \qquad C_1 = 1 + \frac{4-e^2}{e^2-e^{-2}} = \frac{e^2 - e^{-2} + 4 - e^2}{e^2-e^{-2}}$$

$$= \underline{\frac{4-e^{-2}}{e^2-e^{-2}}}$$

So in total:

$$x^*(t) = \frac{4-e^{-2}}{e^2-e^{-2}} e^t - \frac{4-e^2}{e^2-e^{-2}} e^{-t}$$

$$u^*(t) = \frac{4-e^{-2}}{e^2-e^{-2}} e^t + \frac{4-e^2}{e^2-e^{-2}} e^{-t}$$

Now (1) $U = \mathbb{R}$ is convex

$H''_{xx} = -2$

$H''_{xu} = 0$ $\qquad H(H) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $\qquad \Delta_1: \; -2 \quad -2 < 0$

$H''_{uu} = -2$ $\qquad \Delta_2 = 4 > 0$

so (2) H is concave in (x,u)

Thus the Mangasarian tells us that (x^*, u^*) is the optimal pair.

NOTE: Could write this as $u' = x$, which when differentiated $x' = u'' = u$

so $u'' - u = 0$ and $r^2 = 1$ $\quad r = \pm 1$

$u(t) = C_1 e^t + C_2 e^{-t}$

and $x(t) = u'(t) = C_1 e^t - C_2 e^{-t}$

(So the diagonalization was not necessary here.)