Lecture 8; Optimal control theory : Continuous time

12/9-22

≈ Ch. 9

GOAL: Optimization (max / min) of functionals with constraints w.r.t. some control function.

- What is a functional?

A mapping from a space X into the real (or complex) numbers.

- X can be a space of functions!

EX: Let $I = [0, 1]$, then $C(I, \mathbb{R})$ is the set of all continuous functions from I into $\mathbb{R}$.

$f(x) = x$, $x \in [0,1]$ is s.t. $f \in C(I, \mathbb{R})$

Let $X := C(I, \mathbb{R})$.

An example of a functional from X into $\mathbb{R}$ is the integral:

$$J(f) := \int_0^1 f(t)\, dt \in \mathbb{R}$$

$\in X = C(I, \mathbb{R})$

For our $f(x) = x \in$,

$$J(f) = \int_0^1 x\, dx = \frac{1}{2}(1-0) = \frac{1}{2} \in \mathbb{R}$$

So: $J : C(I, \mathbb{R}) \to \mathbb{R}$ is a functional

Optimal control theory, std. problem

$$(*)\begin{cases} \max\limits_{u} \displaystyle\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ \text{s.t.} \quad x'(t) = g(t, x(t), u(t)) \\ \qquad x(t_0) = x_0 \end{cases}$$

An ODE for $x(t)$, depends on $u(t)$

Either: $\underbrace{x(t_1) = x_1}_{\text{may have a fixed terminal condition}}$ or $\underbrace{x(t_1) \text{ free}}_{\text{no terminal condition}}$

$u(t) \in \underbrace{\mathcal{U}}_{\text{control region}} \subseteq \mathbb{R}, \ \forall t$

- $u(t)$; control. We choose this variable.
- $x(t)$; the state variable. Determined via ODE.
- Omit t for notational simplicity: $x = x(t)$, $u = u(t)$

NOTE: Choose u $\Rightarrow$ Get x from ODE.

How to solve this problem?

$\Rightarrow$ Introduce Hamiltonian:

(9.4.1) $\mathcal{H} = \mathcal{H}(t, x, u, p) := p_0 f(t,x,u) + p\, g(t,x,u)$

the adjoint function

where $p = p(t)$ and $p_0 = 0$ or $p_0 = 1$

usually $p_0 = 1$

Thm (Pontryagin's max. principle, necessary cond.)

If (x^*, u^*) is optimal $(\star)$, then there exists a cont. function $p(t)$ and a number p_0 (0 or 1) s.t. $\forall\ t \in [t_0, t_1]$:

- $(p_0, p(t)) \neq (0,0)$ and

(A) u^* maximizes $\mathcal{H}(t, x^*, u, p)$ wrt. u.

(B) $$p'(t) = -\frac{\partial \mathcal{H}(t, x^*, u^*, p)}{\partial x}$$

(C) where $x(t_1) = x_1 \Rightarrow$ no condition

$x(t_1)$ free $\Rightarrow p(t_1) = 0$

Thm (Mangasarian, sufficient condition):

Assume (x^*, u^*) satisfies (A)-(B)-(C) with $p_0 = 1$.

Suppose $\mathcal{U}$ is convex and

$u, \tilde{u} \in \mathcal{U}$, $\lambda \in [0,1]$, then $\lambda u + (1-\lambda)\tilde{u} \in \mathcal{U}$

$$(x, u) \longmapsto \mathcal{H}(t, x, u, p)$$

is concave for every $t \in [t_0, t_1]$. Then, (x^*, u^*) is optimal for $(\star)$.

METHOD FOR FINDING OPTIMAL CONTROL

For $p_0 = 0, 1$:

1) Maximize $\mathcal{H}$ w.r.t. u: $\dfrac{\partial \mathcal{H}}{\partial u} = 0$ (For (A) to hold)

2) Solve adjoint diff. eq:

$$p'(t) = -\frac{\partial \mathcal{H}(t, x^*, u^*, p)}{\partial x} \quad \text{for } p$$

(For (B) to hold)

3) → Take terminal condition for p into account.
→ Use this to solve for $u^*(t)$. (For (C) to hold)
→ Given $u^*(t)$, use ODE from (*) to solve for $x^*(t)$.

4) Make sure $\mathcal{U}$ is convex and $\mathcal{H}$ is concave in (x, u) (For Mangasarian)

$$\Downarrow$$

(u^*, x^*) maximize (A) and hence are the optimal control and controlled process.

Ex: $\max_u \int_0^T (1 - tx - u^2)\,dt$

$$(\Delta)\begin{cases} \text{s.t.} & x' = u \\ & x(0) = x_0 \\ & \mathcal{U} = \mathbb{R} \quad (\text{is convex}) \end{cases}$$

Let T, x_0 are given

Hamiltonian:

$$\mathcal{H}(t, x, u, p) = p_0\,(1 - tx - u^2) + pu$$

2 cases: $p_0 = 0$ and $p_0 = 1$

$p_0 = 0$: $\mathcal{H} = pu$

NOTE: For (B) to hold

$$p'(t) = -\mathcal{H}'_x = 0 \Rightarrow p(t) = \underbrace{C}_{\text{some constant}} \in \mathbb{R}$$

But $\mathcal{H} = Cu$ has no maximizer in $\mathcal{U}$ (since $\mathcal{U} = \mathbb{R}$). Hence, no solutions when $p_0 = 0$.

(A) can't hold

$p_0 = 1$: $\mathcal{H} = (1 - tx - u^2) + pu$

METHOD:

1) For (A): $\dfrac{\partial \mathcal{H}}{\partial u} = -2u + p$

FOC: $-2\hat{u} + p = 0$

$$\hat{u} = \frac{p}{2}$$

candidate optimal control

2) For (B): $p'(t) = -\mathcal{H}'_x = t$

$$p(t) = \int t \, dt = \frac{1}{2} t^2 + C$$

(general solution of the ODE for p)

3) For (C): $x(T)$ free $\Rightarrow$ $p(T) = 0$

$$\underbrace{\frac{1}{2} T^2 + C}_{p(T)} = 0$$

$$\Downarrow$$

$$C = -\frac{1}{2} T^2$$

So: $p(t) = \frac{1}{2} t^2 - \frac{1}{2} T^2$ (particular solution of ODE for p)

Hence: $\hat{u}(t) = \frac{1}{2} p(t) = \frac{1}{4}(t^2 - T^2)$

From 1)

Need to find $\hat{x}(t)$:

From ODE in original problem:

$$\hat{x}'(t) = \hat{u}(t) = \frac{1}{4}(t^2 - T^2)$$

$$\hat{x}(t) = \int \frac{1}{4}(t^2 - T^2)\, dt = \frac{1}{4}\left(\frac{1}{3} t^3 - T^2 t\right) + C$$

$$= \frac{1}{12} t^3 - \frac{1}{4} T^2 t + C$$ (general solution of ODE)

Know: $\hat{x}(0) = x_0 = C$

From ODE in

$$\Downarrow$$

$$\hat{x}(t) = \frac{1}{12} t^3 - \frac{1}{4} T^2 t + x_0$$

4) $\underline{\mathcal{U} \text{ is convex}}$: $\mathcal{U} = \mathbb{R}$, and $\mathbb{R}$ is convex

$\left(x, y \in \mathbb{R},\ \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in \mathbb{R} \right)$

To check concavity of $\mathcal{H}$: $\quad (x, u) \rightarrow \mathcal{H}$

Need to find Hessian matrix, i.e..

$\mathcal{H}'_x = -t \qquad\qquad \mathcal{H}''_{xx} = 0$

$\mathcal{H}'_u = -2u + p \qquad\qquad \mathcal{H}''_{uu} = -2$

$\mathcal{H}''_{xu} = \mathcal{H}''_{ux} = 0$

$\Downarrow$

Hessian matrix:

$$H = \begin{array}{cc} \mathcal{H}''_{xx} & \mathcal{H}''_{xu} \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}} \\ \mathcal{H}''_{ux} & \mathcal{H}''_{uu} \end{array}$$

Thm. 2.33: $\mathcal{H}$ concave $\Leftrightarrow (-1)^r \Delta_r(\vec{x}) \geq 0 \ \forall$

$\vec{x} = (x, u)$

and all $\Delta_r(\vec{x})$, $r = 1, \ldots, \underbrace{n}_{2}$,

principal minors.

Here: $n = 2$ (2×2 matrix H)

$r=1$: $\quad \underbrace{\Delta_{1,1}}_{\text{delete row 1 column 1}} = -2 \quad , \quad \underbrace{\Delta_{1,2}}_{\text{delete row 2, column 2}} = 0$

$(-1)^1(-2) = 2 \geq 0 \qquad (-1)^1 0 = 0 \geq 0$

$r=2$: $\quad \Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} = 0 \ ; \ (-1)^2 \cdot 0 = 0 \geq 0$

$\Rightarrow \mathcal{H}$ is concave in (x, u).

Hence, $(u^*, x^*) = (\hat{u}, \hat{x})$

$$= \left(\frac{1}{2}t^2 - \frac{1}{4}T^2,\ \underline{\underline{\frac{1}{12}t^3 - \frac{1}{4}T^2 t + x_0}}\right)$$

Problem 1

$$(\sim)\begin{cases} \max \int_0^2 (3 - x^2 - u^2)\,dt \\ \text{s.t.} \quad x' = u \\ \quad x(0) = 1 \\ \quad x(2) = 4 \\ \quad \mathcal{U} = \mathbb{R} \quad (\text{convex}) \end{cases}$$

Hamiltonian:

$$\mathcal{H} = p_0(3 - x^2 - u^2) + pu$$

2 cases:

$p_0 = 0$: Same argument as before: $\mathcal{H}$ has no maximizers in $\mathcal{U}$ $\Rightarrow$ no solutions for $p_0 = 0$.

$p_0 = 1$: $\mathcal{H} = 3 - x^2 - u^2 + pu$

1) For (A): $\frac{\partial \mathcal{H}}{\partial u} = -2u + p$

Set equal 0: $-2\hat{u} + p = 0$

$$\hat{u} = \frac{p}{2}$$

2) For (B): $\quad p'(t) = -\mathcal{H}'_x = 2x$

Note that: $\boxed{\hat{u}'(t) = \frac{1}{2} p'(t) = \frac{1}{2} 2\hat{x}(t) = \hat{x}(t)}$

(From 1))

Also, from diff. eq. in $(\sim)$;

$$\boxed{\hat{x}'(t) = \hat{u}(t)}$$

$\hat{u}'' = \hat{x}' = \hat{u}$

$\hat{u}'' - \hat{u} = 0$

Sec. 6.2

This is a system of linear diff. eq:

$$\begin{bmatrix} \hat{u} \\ \hat{x} \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \hat{u} \\ \hat{x} \end{bmatrix}$$

Sec: 6.5