<u>Solution:</u> Exam 09/2014 DRE 7017

<u>1.</u>

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\qquad \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

b) $\underline{z}' = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix} \underline{z}$ with $\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \underline{\begin{pmatrix} x+1 \\ y-2 \end{pmatrix}}$

$$\underline{z} = c_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t}$$

$$\begin{vmatrix} 7-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 8\lambda - 9 = 0$$

$$\lambda_1 = \underline{9} \qquad \lambda_2 = \underline{-1}$$

$\underline{\lambda_1 = 9}$: $\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \underline{v} = \underline{0}$ $\qquad \underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1}$: $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \underline{v} = \underline{0}$ $\qquad \underline{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\underline{z} = c_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{9t} + c_2 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{9t} + c_2 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$x = 2c_1 e^{9t} + c_2 e^{-t} + 1$$

$$y = c_1 e^{9t} - 2c_2 e^{-t} - 2$$

$\lambda_1 > 0$ and $\lambda_2 < 0$:

$$\left.\begin{array}{l} e^{at} \to \infty \\ e^{-t} \to 0 \end{array}\right\} \text{ as } t \to \infty$$

Choose $c_1 = 0$, c_2 arbitrary:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2 \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \text{ as } t \to \infty$$

For all other values ($c_1 \neq 0$), $\begin{pmatrix} x \\ y \end{pmatrix}$ will <u>not</u> approach $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$ as $t \to \infty$.

Since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 + 1 \\ c_1 - 2c_2 - 2 \end{pmatrix}$, the condition $c_1 = 0$ means that

$$\left.\begin{array}{l} x_0 = c_2 + 1 \\ y_0 = -2c_2 - 2 \end{array}\right\} \quad \text{or} \quad \begin{array}{l} x_0 - 1 = c_2 \\ y_0 + 2 = -2c_2 \end{array}$$

$$\Updownarrow$$

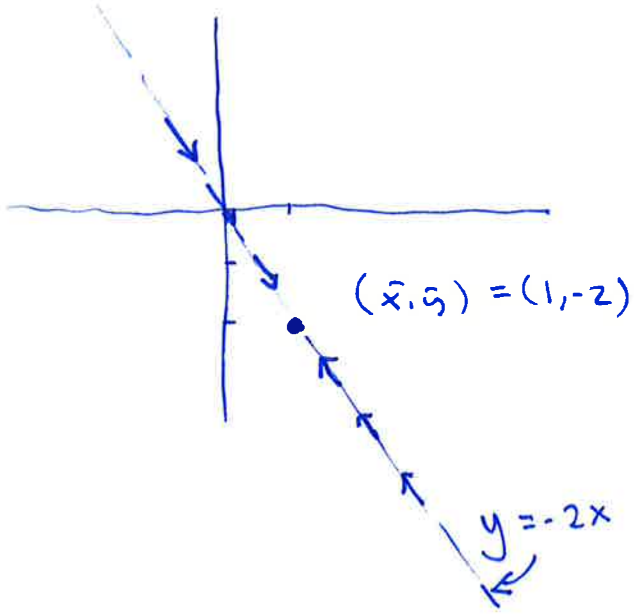
$$y_0 + 2 = -2(x_0 - 1)$$

$$\Updownarrow$$

$$y_0 = -2x_0$$

Conclusion:

$(x, y) \to (\bar{x}, \bar{y})$ as $t \to \infty$ for all initial states (x_0, y_0) such that $\underline{y_0 = -2x_0}$



2. $f = x^2 + y^2 + z^2 + w^2 + xw - yz$

a) $H(t) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

$$\left.\begin{aligned} D_1 &= \underline{2} \\ D_2 &= \underline{4} \\ D_3 &= 2\cdot(4-1) = \underline{6} \\ D_4 &= 2\cdot D_3 - 1\cdot 1\cdot(4-1) = 12-3 = \underline{9} \end{aligned}\right\}$$

$H(t)$ pos. defn.

f $\underline{\text{convex}}$

$\underline{\text{not concave}}$

b) Stationary pts:

$$\begin{aligned} f'_x &= 2x + w = 0 \\ f'_y &= 2y - z = 0 \\ f'_z &= 2z - y = 0 \\ f'_w &= 2w + x = 0 \end{aligned}$$

$y = z = 0$

$x = w = 0$

Stat. pts:

$(x, y, z, w) = \underline{(0, 0, 0, 0)}$

Global min: $\underline{(0,0,0,0)}$ with min. value $\underline{f=0}$

Global max: no global max

(since $H(t)(\underline{x})$ is pos. defn. for all $\underline{x}$)

(Alt: $f = \frac{1}{2}x^2 + \frac{1}{2}w^2 + \frac{1}{2}(x+w)^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + \frac{1}{2}(y-z)^2$)

3. $\max \int_0^T \ln(ax - bu)\,dt$ $\quad \begin{cases} x_0 = 1 \\ x_T = \tfrac{1}{2} e^{aT/b} \\ x' = u \\ u \in U = \mathbb{R} \end{cases}$

a) $F = \ln(ax - bu)$

$$\left.\begin{aligned} F'_x &= \frac{a}{ax-bu} \\ F'_u &= \frac{-b}{ax-bu} \end{aligned}\right\} \quad H(F) = \begin{pmatrix} \dfrac{-a\cdot a}{(ax-bu)^2} & \dfrac{-a(-b)}{(ax-bu)^2} \\ \dfrac{b\cdot a}{(ax-bu)^2} & \dfrac{+b(-b)}{(ax-bu)^2} \end{pmatrix}$$

$$= \frac{1}{(ax-bu)^2} \cdot \begin{pmatrix} -a^2 & ab \\ ab & -b^2 \end{pmatrix}$$

F is concave in (x,u) $\quad \begin{cases} D_1 = -a^2/(ax-bu)^2 \le 0 \quad \Delta_1 = -b^2/(ax-bu)^2 \le 0 \\ D_2 = 0 \end{cases}$

b) Alt I: Euler eqn. $\quad \max \int_0^T \ln(ax - b\dot{x})\,dt \;,\; \begin{matrix} x_0 = 1 \\ x_T = \tfrac{1}{2} e^{aT/b} \end{matrix}$

$$F'_x - \frac{d}{dt} F'_{\dot{x}} = \frac{a}{ax - b\dot{x}} - \frac{-b(-1)}{(ax-b\dot{x})^2} \cdot (a\dot{x} - b\ddot{x}) = 0$$

$$a(ax - b\dot{x}) - b(a\dot{x} - b\ddot{x}) = 0$$

$$b^2 \ddot{x} - 2ab\dot{x} + a^2 x = 0$$

Char. eqn: $b^2 r^2 - 2ab\,r + a^2 = 0$

$$r = \frac{2ab \pm \sqrt{(2ab)^2 - 4 \cdot b^2 \cdot a^2}}{2b^2} = \frac{a}{b} \quad \text{(double root)}$$

$$x = C_1 \cdot e^{a/b\,t} + C_2 t e^{a/b\,t}$$

$x(0) = 1$: $\quad 1 = C_1 + C_2 \cdot 0 = C_1 \Rightarrow C_1 = 1$

$x(T) = \frac{1}{2} e^{aT/b}$: $\quad \frac{1}{2} e^{aT/b} = e^{aT/b}(1 + C_2 T) \Rightarrow C_2 T = -1/2$

$$C_2 = -\frac{1}{2T}$$

$\Rightarrow x = \underline{\left(1 - \frac{1}{2}\frac{t}{T}\right) e^{a/b\,t}}$ $\quad$ (gives max since F is concave in (x,u))

Alt 2: Hamiltonian / maximum principle

$$H = p_0 \cdot \ln(ax - bu) + pu$$

$p_0 = 0$: $\quad H = pu$

(B) $\dot{p} = -H'_x = 0 \Rightarrow p = C \neq 0$ (const.)

$H = pu = C \cdot u$ has no max for $u \in \mathbb{R}$

$\Downarrow$

no candidate for max with $p_0 = 0$

$p_0 = 1$: $\quad H = \ln(ax - bu) + pu$

(A) $H'_u = \dfrac{-b}{ax - bu} + p = 0$ (stationary pts are only max since H is concave in (x, u))

$\Updownarrow$

$ax - bu = \dfrac{b}{p}$ (and $p \neq 0$)

(B) $\dot{p} = -H'_x = -\dfrac{a}{ax - bu} = -\dfrac{a}{b} \cdot p$ (from A)

$\dot{p} = \dfrac{a}{b} p \Rightarrow p(t) = C \cdot e^{-a/b\, t} \Rightarrow ax - bu = \dfrac{b}{Ce^{-a/b\, t}}$

$$ax - b\dot{x} = \frac{b}{c} e^{a/b\, t}$$

(c) no cond.

First order linear diff. eqn: $x = x_h + x_p$

x_h: $ax - b\dot{x} = 0$

$a - br = 0$ (char. eqn.) $\Rightarrow r = a/b \Rightarrow x_h = De^{\frac{a}{b}t}$

x_p: Try $Ee^{a/b\, t}$: doesn't work

Try $(E + Ft)e^{a/b\, t}$: gives

$$a \cdot (E + Ft)e^{a/b\, t} - b \cdot \left(F + (E + Ft) \cdot \frac{a}{b}\right)e^{\frac{a}{b}t} = \frac{b}{c} e^{\frac{a}{b}t}$$

$$a\cancel{(E + Ft)} - bF - a\cancel{(E + Ft)} = \frac{b}{c}$$

$F = -1/c \quad E$ arbitrary (can go into x_h)

$$x = x_h + x_p = De^{\frac{a}{b}t} + \left(-\frac{1}{c}t\right)e^{a/b\,t} = \left(D - \frac{1}{c}t\right)e^{\frac{a}{b}t}$$

$x(0) = 1$: $\quad 1 = D \cdot e^0 = D \Rightarrow D = 1$

$x(T) = \frac{1}{2}e^{aT/b}$: $\quad \frac{1}{2}e^{aT/b} = \left(1 - \frac{1}{c}T\right)e^{aT/b}$

$$\frac{1}{2} = 1 - T/c \Rightarrow T/c = 1/2 \Rightarrow c = 2T$$

$$\underline{x = \left(1 - \frac{1}{2}\frac{t}{T}\right)e^{at/b}}$$

(gives max since F is concave in (x, u))

4. $f_n = \frac{x^n}{n}$ in $C([0,1])$

a) $f_1 = x \quad f_2 = x^2/2$

$\|f_1\| = \sup_x |x| = 1$

$\|f_2\| = \sup_x |x^2/2| = 1/2$

$d(f_1, f_2) = \sup_x |x - x^2/2| = 1 - 1/2 = 1/2$

sup = max and is attained at $x=1$ in all cases.

b) $f_n = x^n/n \quad f_{n+k} = x^{n+k}/n+k$

$$d(f_n, f_{n+k}) = \sup \left| \frac{x^n}{n} - \frac{x^{n+k}}{n+k} \right| = \frac{1}{n} - \frac{1}{n+k}$$

(since $\left(x^n/n - x^{n+k}/n+k\right)' = x^{n-1} - x^{n+k-1} > 0$ in $(0,1)$)

$$d(f_n, f_{n+k}) = \frac{1}{n} - \frac{1}{n+k} = \frac{n+k-n}{n\cdot(n+k)} = \frac{k}{n\cdot(n+k)}$$

c) (f_n) is Cauchy with limit $f=0$.

Proof.

Alt. I: Since $\|f_n\| = \sup_x |x^n/n| = 1/n$, we have

$$\|f_n - f\| = \|f_n\| = 1/n \to 0$$

with $f=0$, so $f_n \to f = 0$ as $n \to \infty$

Alt. 2: $d(f_n, f_{n+k}) = \frac{k}{n\cdot(n+k)} = \frac{1}{n} \cdot \frac{k}{n+k} < \frac{1}{n}$ for all $n, k \geq 1$

$\Downarrow$

for any $\varepsilon > 0$, there is an N such that

$n, m \geq N \implies \|f_n - f_m\| < \varepsilon$

(we can take $N > 1/\varepsilon$).

The limit $f=0$: see alt. I.

Then (f_n) is Cauchy and $(f_n) \to f$ since

$\longrightarrow$ $C[0,1]$ is ~~bounded~~ complete ($[0,1]$ compact)