Solutions	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 2nd, 2018 at 0900 - 1200

Question 1.

(a) We have that

$$A+I=\begin{pmatrix}1&1&0\\0&1&1\\1&1&0\end{pmatrix}\quad\Rightarrow\quad(A+I)^2=\begin{pmatrix}1&2&1\\1&2&1\\1&2&1\end{pmatrix}$$

This means that $(A+I)^2\cdot\mathbf{v}=\mathbf{0}$ is given by $v_1+2v_2+v_3=0$, with v_2,v_3 as free variables, and $v_1=-2v_2-v_3$ basic. The solutions can be written as $\text{span}(\mathbf{w}_1,\mathbf{w}_2)$, since

$$\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}-2v_2-v_3\\v_2\\v_3\end{pmatrix}=v_2\begin{pmatrix}-2\\1\\0\end{pmatrix}+v_3\begin{pmatrix}-1\\0\\1\end{pmatrix}=v_2\mathbf{w}_1+v_3\mathbf{w}_2$$

It is clear that $\mathbf{w}_1$ is not an eigenvector for A, since $A\mathbf{w}_1=\begin{pmatrix}1&0&-1\end{pmatrix}^T\neq\lambda\mathbf{w}_1$ for any λ.

(b) Clearly, $\lambda_1=-1$ is an eigenvalue since $\det(A+I)=0$. We find the other eigenvalues by $\lambda_1+\lambda_2+\lambda_3=\text{tr}(A)=-1$ and $\lambda_1\lambda_2\lambda_3=\det(A)=1$, which gives $\lambda_2+\lambda_3=0$ and $\lambda_2\lambda_3=-1$, or $\lambda_2=-1$ and $\lambda_3=1$. Alternatively, the characteristic equation is

$$\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\1&1&-1-\lambda\end{vmatrix}=0$$

This gives $-\lambda(-\lambda(-1-\lambda)-1)-1(-1)=\lambda^2(-\lambda-1)+\lambda+1=(\lambda+1)(-\lambda^2+1)=0$ by cofactor expansion along the first row, or $(1+\lambda)(1-\lambda)(1+\lambda)=0$. The eigenvalues are $\lambda_1=\lambda_2=-1$ and $\lambda_3=1$. We have that $E_{-1}=\text{span}(\mathbf{w})$ by a direct computation: Using the matrix $A+I$ above, we obtain that $v_2+v_3=0$, or $v_2=-v_3$, and $v_1+v_2=0$, or $v_1=-v_2=v_3$, with v_3 free, and

$$\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}v_3\\-v_3\\v_3\end{pmatrix}=v_3\begin{pmatrix}1\\-1\\1\end{pmatrix}=v_3\mathbf{w}$$

with $\mathbf{w}=\mathbf{w}_2-\mathbf{w}_1$. Since E_{-1} has dimension 1, when $\lambda=-1$ has multiplicity 2, A is not diagonalizable.

(c) An eigenvector $\mathbf{v}$ satisfies $(A-\lambda I)\mathbf{v}=\mathbf{0}$, and is therefore a generalized eigenvector with $n=1$. We see that there is an eigenvector $\mathbf{w}_3$ such that $E_1=\text{span}(\mathbf{w}_3)$, since $\lambda=1$ has multiplicity 1. Since eigenvectors are generalized eigenvectors, the set $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ is a set of three generalized eigenvectors for A. Explicitly, we may chose

$$\mathbf{w}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad\Rightarrow\quad\begin{vmatrix}-2&-1&1\\1&0&1\\0&1&1\end{vmatrix}=2+2=4\neq0$$

and the non-zero determinant proves that $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ are linearly independent generalized eigenvectors for A.

Question 2.

(a) The set D is convex since it is the half-plane $x\geq0$, with a straight line as the boundary. Explicitly, if two points P,Q in D, such that $x_P,x_Q\geq0$, then all points in the line segment $[P,Q]$ satisfy $x\geq0$ as well. To check if f is concave, we compute

$$f'_x=\frac{1}{2}x^{-1/2}y=\frac{y}{2\sqrt{x}}\quad\Rightarrow\quad f''_{xx}=-\frac{1}{4}x^{-3/2}y=\frac{-y}{4x\sqrt{x}}$$

This means that the first principal minor $D_1=f''_{xx}$ of $H(f)$ can take both positive and negative values on D. For example, $D_1(1,1)=-1/4$ and $D_1(1,-1)=1/4$. Therefore, f is not concave.

(b) The set $f(D) = \mathbb{R}$ since $f(1, a) = a$ and $(1, a) \in D$ for any real number a. This is not a compact set, since it is not bounded.

(c) We have that the set $U_f(a)$ is given by the inequality $\sqrt{x} \cdot y \geq a$, and this can be written as $y \geq a/\sqrt{x}$ when $x \neq 0$. The function $f(x) = a/\sqrt{x}$ has derivatives

$$f' = \frac{-a}{2x\sqrt{x}} \quad \Rightarrow \quad f'' = \frac{3a}{2x^2\sqrt{x}} > 0$$

If $a > 0$, then $x > 0$, and f is convex. This implies that $U_f(a)$ is a convex set. If $a \leq 0$, then $U_f(a)$ is the union of the y-axis, and the set of points over the graph of f when $x > 0$. In this case, $U_f(a)$ is not a convex set. For example, $P = (2, 0)$ and $Q = (0, 2a - 2)$ are in $U_f(a)$ when $a < 0$, but the midpoint $(1, a - 1)$ on the line segment $[P, Q]$ is not in $U_f(a)$ since $\sqrt{1} \cdot (a - 1) < a$. We conclude that $U_f(a)$ is a convex set if and only if $a > 0$.

(d) Let E be the set of points satisfying the constraint $x^2 + y^2 \leq 2x$. This is a compact set since $x^2 - 2x + y^2 \leq 0$ gives

$$x^2 - 2x + 1 + y^2 \leq 1 \quad \Rightarrow \quad (x - 1)^2 + y^2 \leq 1$$

which is a circle with center $(1, 0)$ of radius 1, and the points inside that circle. Hence f attains a max and a min on E. Since $(x, y) \in E$ if and only if $(x, -y) \in E$, and $f(x, -y) = -f(x, y)$, it follows that the values attained are in the interval $[-M, M]$ for some number M. Moreover, M is attained for $y > 0$, and $-M$ for $y < 0$. To find M, we consider the max problem, after using $\phi(x) = x^2$ on the objective function f, to simplify the computations (which we can do, since ϕ is monotonously increasing when $y > 0$). We obtain the problem

$$\max xy^2 \text{ when } x^2 + y^2 - 2x \leq 0$$

This is a Kuhn-Tucker problem in standard form. We write $\mathcal{L} = x^2y - \lambda(x^2 + y^2 - 2x)$, and obtain the first order conditions

$$\mathcal{L}'_x = y^2 - \lambda(2x - 2) = 0, \quad \mathcal{L}'_y = 2xy - \lambda(2y) = 0$$

together with the constraint $x^2 + y^2 - 2x \leq 0$ and the complementary slackness condition $\lambda \geq 0$ and $\lambda(x^2 + y^2 - 2x) = 0$. When $\lambda = 0$, we get $y = 0$ and $0 < x < 2$, which gives a condidate point with $xy^2 = 0$. When $\lambda > 0$, we get $y = 0$ or $x = \lambda$, and $y = 0$ gives $\lambda = 0$, a contradiction. Therefore, $x = \lambda$, which gives $y^2 = x(2x - 2) = 2x^2 - 2x$ and $x^2 + (2x^2 - 2x) - 2x = 0$, or $3x^2 - 4x = 0$. This gives $x = 0$ or $x = 4/3$, and $x = 0$ gives $\lambda = 0$, a contradiction. Therefore, $x = 4/3$, $y = \pm\sqrt{8/9} = \pm 2\sqrt{2}/3$, and $\lambda = x = 4/3$. At these candidate points, we have that $xy^2 = 32/27$. This solves the maximum problem (there are no points in E where NDCQ fails, since E is bounded by a circle), and therefore we have that $M = \sqrt{32/27} = 4\sqrt{6}/9$. It means that the original minimum problem has minimum value $-M = -4\sqrt{6}/9$, and the minimizer is $(x, y) = (4/3, -2\sqrt{2}/3)$.

Question 3.

(a) Let us write $A = (\mathbf{v}_1|\mathbf{v}_2|\dots|\mathbf{v}_n)$. Since $\mathbf{x} \in \Delta_{n-1}$, we have that $x_i \geq 0$ and $x_1 + \cdots + x_n = 1$. We consider the vector

$$\mathbf{y} = A \cdot \mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

and must show that $\mathbf{y} \in \Delta_{n-1}$. Since $a_{ij} \geq 0$ and $x_i \geq 0$, it follows that each component y_i of $\mathbf{y} = A \cdot \mathbf{x}$ is non-negative, or $y_i \geq 0$. Moreover, since each $\mathbf{v}_i$ has column sum 1, and $x_1 + \cdots + x_n = 1$, it follows that $\mathbf{y} = A\mathbf{x}$ has column sum $y_1 + \cdots + y_n = 1$. This means that $\mathbf{y} \in \Delta_{n-1}$, and the map is well-defined.

(b) It is clear that Δ_{n-1} is compact: It is clearly closed, since it is given by equations and closed inequalities, and it is bounded since $0 \leq x_i \leq 1$ for all i. It is non-empty as well, since $(1, 0, 0, \dots, 0) \in \Delta_{n-1}$. We need to show that it is convex to apply Brouwer's result: When $\mathbf{x}, \mathbf{y} \in \Delta_{n-1}$, we have that all points on the line segment $[\mathbf{x}, \mathbf{y}]$ are given by

$$\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \Delta_{n-1}$$

with $0 \leq \lambda \leq 1$, and all point of this form is in Δ_{n-1} since $z_i = \lambda x_i + (1 - \lambda)y_i \geq 0$ and $z_1 + \cdots + z_n = \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1$. This shows that Δ_{n-1} is convex, and it follows by Brouwer's fixed point theorem that $A : \Delta_{n-1} \to \Delta_{n-1}$ has a fixed point.

(c) A fixed point is an eigenvector with $\lambda = 1$, so we compute E_1 for the given matrix. This gives the echelon form

$$A - I = \begin{pmatrix} -1 & 0.6 & 0.5 \\ 0.7 & -0.6 & 0.5 \\ 0.3 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0.6 & 0.5 \\ 0 & -0.18 & 0.85 \\ 0 & 0.18 & 0.85 \end{pmatrix} \to \begin{pmatrix} -1 & 0.6 & 0.5 \\ 0 & -0.18 & 0.85 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence x_3 is free, $-0.18x_2 + 0.85x_3 = 0$, or $x_2 = 85x_3/18$, and $-x_1 + 0.6x_2 + 0.5x_3 = 0$, or $x_1 = 0.6(85x_3/18) + 0.5x_3 = 10x_3/3$. The eigenvectors in E_1 that are also in Δ_2 must satisfy

$$x_1 + x_2 + x_3 = 1 \quad \Rightarrow \quad \left(\frac{10}{3} + \frac{85}{18} + 1\right) x_3 = \frac{60 + 85 + 18}{18} \cdot x_3 = \frac{163}{18} \cdot x_3 = 1$$

This gives $x_3 = 18/163$, and the fixed point is therefore given by

$$(x_1, x_2, x_3) = \left(\frac{60}{163}, \frac{85}{163}, \frac{18}{163}\right)$$