EVALUATION GUIDELINES - Written examination

DRE 70171
Mathematics, Ph.D.

Department of Economics

Start date:	13.10.2020	Time 09:00
Finish date:	13.10.2020	Time 12:00

For more information about formalities, see examination paper.

Solutions	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 13th, 2020 at 0900 - 1200

Question 1.
We compute the inner product, given by the integral

$$f \cdot g = \int_0^1 t^{a+b}\,\mathrm{d}t = \left[\frac{1}{a+b+1}t^{a+b+1}\right]_0^1 = \frac{1}{a+b+1}$$

Question 2.

a) We have that $f'_x = 2x + \sqrt{12}\,z$, $f'_y = 4y$, and $f'_z = 6z + \sqrt{12}\,x$, and the stationary points of f are given by the first order conditions

$$2x + \sqrt{12}\,z = 0, \quad 4y = 0, \quad 6z + \sqrt{12}\,x = 0$$

This gives $x = -\sqrt{3}z$, $y = 0$, and that z is free, and $(x, y, z) = (-\sqrt{3}\,t, 0, t)$ for $t \in \mathbb{R}$ are therefore the stationary points of f. The Hessian matrix of f is given by

$$H(f) = \begin{pmatrix} 2 & 0 & \sqrt{12} \\ 0 & 4 & 0 \\ \sqrt{12} & 0 & 6 \end{pmatrix}$$

and we compute its principal minors:

$$\begin{aligned} \Delta_1&: 2,\ 4,\ 6 \quad > 0, \\ \Delta_2&: 8,\ 0,\ 24 \quad \geq 0, \\ \Delta_3&: 0 \end{aligned}$$

This means that f is convex, and the stationary points $(x, y, z) = (-\sqrt{3}\,t, 0, t)$ for $t \in \mathbb{R}$ are minimum points of f, with $f(-\sqrt{3}\,t, 0, t) = 0$ for all t.

b) The Lagrangian is $\mathcal{L} = x^2 - y^2 - x^3 - \lambda(x^2 + y^2)$, and the Kuhn–Tucker conditions for (x^*, y^*) are:

$$\begin{aligned} \mathcal{L}'_x &= 2x - 3x^2 - 2\lambda x = x(2 - 3x - 2\lambda) = 0 \\ \mathcal{L}'_y &= -2y - 2\lambda y = -2y(1 + \lambda) = 0 \\ C&: x^2 + y^2 \leq 1 \\ CSC&: \lambda \geq 0 \text{ and } \lambda \cdot (x^2 + y^2 - 1) = 0 \end{aligned}$$

Suppose first that $\lambda = 0$. Then the system is

$$x(2 - 3x) = 0, \quad -2y = 0, \quad x^2 + y^2 \leq 1$$

and the candidate points are $(0, 0)$ and $(2/3, 0)$ with $\lambda = 0$. Next, suppose that $\lambda > 0$. Then the system is

$$x(2 - 3x - 2\lambda) = 0, \quad -2y(1 + \lambda) = 0, \quad x^2 + y^2 = 1$$

Then $y = 0$ by the middle equation, $x = \pm 1$ by the last equation, and $\lambda = 1 - 3x/2 = 1 \mp 3/2$. With $x = 1$, we find $\lambda < 0$, so $x = -1$, $y = 0$ and $\lambda = 5/2$ gives the only candidate point $(-1, 0)$. Computing the values of f for the candidate points give:

$$\begin{aligned} f(0, 0) &= 0, \\ f(2/3, 0) &= (2/3)^2 - (2/3)^3 = 4/27 < 1 \\ f(-1, 0) &= 2 \end{aligned}$$

Since the set of points satisfying the constraint $x^2 + y^2 \leq 1$ is compact, the maximal value of $f(x, y)$ on $x^2 + y^2 \leq 1$ is $f_{\max} = 2$ at $(x^*, y^*) = (-1, 0)$ with $\lambda = 5/2$.

Question 3.

a) The steady state solves

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix}2 & 0 & 1\\ 0 & -1 & 0\\ 0 & 2 & 1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\5\\4\end{pmatrix}$$

Performing Gauss–Jordan elimination on the augmented coefficient matrix, we get

$$\left(\begin{array}{ccc|c}2 & 0 & 1 & 2\\ 0 & -1 & 0 & 5\\ 0 & 2 & 1 & 4\end{array}\right) \sim \left(\begin{array}{ccc|c}2 & 0 & 1 & 2\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 14\end{array}\right) \sim \left(\begin{array}{ccc|c}2 & 0 & 0 & -12\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 14\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 0 & -6\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 14\end{array}\right)$$

so the steady state is $(\bar{x}, \bar{y}, \bar{z}) = (-6, -5, 14)$.

b) With $\mathbf{w} = (x+6, y+5, z-14)$ we get

$$\boldsymbol{w}' = \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}2 & 0 & 1\\ 0 & -1 & 0\\ 0 & 2 & 1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\5\\4\end{pmatrix} = \begin{pmatrix}2 & 0 & 1\\ 0 & -1 & 0\\ 0 & 2 & 1\end{pmatrix} \cdot \mathbf{w} = A \cdot \mathbf{w}$$

This means that the matrix A is given by

$$A = \begin{pmatrix}2 & 0 & 1\\ 0 & -1 & 0\\ 0 & 2 & 1\end{pmatrix}$$

To find the eigenvalues of A, we solve the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix}2-\lambda & 0 & 1\\ 0 & -1-\lambda & 0\\ 0 & 2 & 1-\lambda\end{vmatrix} = (2-\lambda)(-1-\lambda)(1-\lambda) = 0$$

so the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = 1$. The base $\{\mathbf{v}_i\}$ of the eigenspace of A for the eigenvalue $\lambda = \lambda_i$ is found by solving the linear system $(A - \lambda I)\,\boldsymbol{v} = \mathbf{0}$. We find

$$E_{\lambda_1}\colon \left(\begin{array}{ccc|c}0 & 0 & 1 & 0\\ 0 & -3 & 0 & 0\\ 0 & 2 & -1 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right), \text{ so } \boldsymbol{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$E_{\lambda_2}\colon \left(\begin{array}{ccc|c}3 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 2 & 2 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & \frac{1}{3} & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right), \text{ so } \boldsymbol{v}_2 = (-3) \cdot \begin{pmatrix}-\frac{1}{3}\\ -1\\ 1\end{pmatrix} = \begin{pmatrix}1\\3\\-3\end{pmatrix}$$

$$E_{\lambda_3}\colon \left(\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & -2 & 0 & 0\\ 0 & 2 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \text{ so } \boldsymbol{v}_3 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

c) The general solution can be found using the above eigenvalues and eigenvectors for A and substituting for $\boldsymbol{w}$:

$$\boldsymbol{w} = C_1 \begin{pmatrix}1\\0\\0\end{pmatrix} e^{2t} + C_2 \begin{pmatrix}1\\3\\-3\end{pmatrix} e^{-t} + C_3 \begin{pmatrix}-1\\0\\1\end{pmatrix} e^{t} \quad \Rightarrow \quad \begin{cases} x(t) &= C_1 e^{2t} + C_2 e^{-t} - C_3 e^{t} - 6\\ y(t) &= 3C_2 e^{-t} - 5\\ z(t) &= -3C_2 e^{-t} + C_3 e^{t} + 14\end{cases}$$

The initial conditions give the system

$$x(0) = C_1 + C_2 - C_3 - 6 = -1, \quad y(t) = 3C_2 - 5 = 10, \quad z(t) = -3C_2 + C_3 + 14 = -1$$

The middle equation gives $C_2 = 5$, and adding the first and last equation gives $C_1 - 2C_2 + 8 = -2$, so $C_1 = 0$. When we substitute this into the first equation, we get $C_3 = 0$. Hence we get

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5e^{-t} - 6\\ 15e^{-t} - 5\\ -15e^{-t} + 14\end{pmatrix} \quad \Rightarrow \quad \lim_{t\to\infty} \begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} = \begin{pmatrix}-6\\-5\\14\end{pmatrix}$$

since $e^{-t} \to 0$ when $t \to \infty$.

Question 4.
By definition, a function f is convex on S if $f(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) \le \lambda f(\boldsymbol{x})+(1-\lambda)f(\boldsymbol{y})$ for λ in $[0,1]$ and $\boldsymbol{x},\boldsymbol{y}$ in S. We assume that $f_1,\dots,f_m$ have this property, and check it for $F = a_1f_1+\cdots+a_mf_m$:

$$\begin{aligned} F(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) &= \sum_{i=1}^m a_i f_i(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) \le \sum_{i=1}^m a_i\left[\lambda f_i(\boldsymbol{x})+(1-\lambda)f_i(\boldsymbol{y})\right] \\ &= \lambda\sum_{i=1}^m a_i f_i(\boldsymbol{x}) + (1-\lambda)\sum_{i=1}^m a_i f_i(\boldsymbol{y}) = \lambda F(\boldsymbol{x})+(1-\lambda)F(\boldsymbol{y}) \end{aligned}$$

We conclude that $f_1,\dots,f_m$ convex and $a_1,\dots,a_m \ge 0$ implies that F is convex.

Question 5.
Since $5-u$ is decreasing on U, we have that $u_3^* = 0$ and

$$J_3(x) = \max_u (5-u)x^2 = 5x^2$$

Using that $J_3(x_3) = 5(u_2x_2)^2$, we have that

$$J_2(x) = \max_u \left\{(5-u)x^2 + 5(ux)^2\right\} = \max_u (5-u+5u^2)x^2$$

Since $h_2(u) = 5-u+5u^2$ is convex on U, with minimum for $u = 1/10$, and $h_2(0) = 5$ and $h_2(1) = 9$, the maximum value is attained for $u_2^* = 1$, hence $J_2(x) = 9x^2$. Using that $J_2(x_2) = 9(u_1x_1)^2$, we have that

$$J_1(x) = \max_u \left\{(5-u)x^2 + 9(ux)^2\right\} = \max_u (5-u+9u^2)x^2$$

Since $h_1(u) = 5-u+9u^2$ is convex on U, with minimum for $u = 1/18$, and $h_1(0) = 5$ and $h_1(1) = 13$, the maximum value is attained for $u_1^* = 1$, hence $J_1(x) = 13x^2$. Using that $J_1(x_1) = 13(u_0x_0)^2$, we have that

$$J_0(x) = \max_u \left\{(5-u)x^2 + 13(ux)^2\right\} = \max_u (5-u+13u^2)x^2$$

Since $h_0(u) = 5-u+13u^2$ is convex on U, with minimum for $u = 1/26$, and $h_0(0) = 5$ and $h_0(1) = 17$, the maximum value is attained for $u_0^* = 1$, hence $J_0(x) = 17x^2$ and the optimal value is $J_0(x_0) = 17x_0^2$.

Question 6.

a) The Lagrangian is $\mathcal{L} = (p_1-x_1)^2 + (p_2-x_2)^2 - \lambda\,(a_1x_1+a_2x_2)$, and we have the following Lagrange conditions:

$$\begin{aligned} \mathcal{L}'_x &= -2(p_1-x_1) - a_1\lambda = 0 \\ \mathcal{L}'_y &= -2(p_2-x_2) - a_2\lambda = 0 \\ C\colon\ & a_1x_1+a_2x_2 = 0 \end{aligned}$$

Using the first two conditions, we get $x_1 = p_1 + \lambda a_1/2$ and $x_2 = p_2 + \lambda a_2/2$. When we put this into the constraint, we get

$$a_1(p_1+\frac{1}{2}\lambda a_1) + a_2(p_2+\frac{1}{2}\lambda a_2) = a_1p_1+a_2p_2+\frac{\lambda}{2}\,(a_1^2+a_2^2) = 0$$

and we can solve this equation for λ since $a_1^2+a_2^2 > 0$:

$$\lambda = -\frac{2}{a_1^2+a_2^2}\,(a_1p_1+a_2p_2)$$

Substituting λ into the expressions for x_1 and x_2 gives

$$x_1 = \frac{(a_2p_1-a_1p_2)}{(a_1^2+a_2^2)}a_2, \quad x_2 = -\frac{(a_2p_1-a_1p_2)}{(a_1^2+a_2^2)}a_1$$

To show that this candidate point is a minimum for f, we observe that f is a positive semi-definite quadratic form and therefore a convex function, and that the constraint is linear. This means that

$$h(x,y) = \mathcal{L}(x,y;\lambda^*)$$

is a convex function when λ^* is the value of λ at the candidate point. By the second order condition, this means that the candidate point (x^*,y^*) found above is a minimum point.

b) The function f is the square of the distance from the given point $\boldsymbol{p}$ to a point $\boldsymbol{x} = (x_1, x_2)$ on the line $a_1x_1 + a_2x_2 = 0$. The point $\boldsymbol{x}^*$ minimizes this function, hence it is the point in the line that is closest to $\boldsymbol{p}$. If the inner product of the two vectors is equal to zero, then they are orthogonal. We compute the inner product of $\boldsymbol{p} - \boldsymbol{x}^*$ and $\boldsymbol{x}^*$ directly, using the notation $c = (a_2p_1 - a_1p_2)$ and $d = (a_1^2 + a_2^2)$:

$$\begin{aligned}(\boldsymbol{p} - \boldsymbol{x}^*) \cdot \boldsymbol{x}^* &= \left(p_1 - a_2\,\frac{c}{d}, p_2 + a_1\,\frac{c}{d}\right) \cdot \left(a_2\,\frac{c}{d}, -a_1\,\frac{c}{d}\right) \\ &= a_2p_1\,\frac{c}{d} - a_2^2\,\frac{c^2}{d^2} - a_1p_2\,\frac{c}{d} - a_1^2\,\frac{c^2}{d^2} \\ &= (a_2p_1 - a_1p_2)\,\frac{c}{d} - (a_1^2 + a_2^2)\,\frac{c^2}{d^2} = \frac{c^2}{d} - \frac{c^2}{d} = 0\end{aligned}$$

This means that the vectors are orthogonal.