EXAMINATION QUESTION PAPER - Written examination

DRE 70171
Mathematics, Ph.D.

Department of Economics

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Examination support materials permitted:	BI-approved exam calculator. Simple calculator. Bilingual dictionary.

Exam	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 13th, 2020 at 0900 - 1200

This exam consists of 10 problems of equal weigth. **You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.**

Question 1.
Let $I = [0, 1]$ be the unit interval, and consider the vector space $V = C(I, \mathbb{R})$ of continuous functions on I, equipped with the inner product

$$f \cdot g = \int_0^1 f(t)g(t)\,\mathrm{d}t$$

Compute $f \cdot g$ when $f(t) = t^a$ and $g(t) = t^b$ for positive constants $a, b > 0$.

Question 2.

a) Find the stationary points of the following function, and classify them:

$$f(x, y, z) = x^2 + 2y^2 + 3z^2 + \sqrt{12}\,xz$$

b) Solve the following Kuhn–Tucker problem:

$$\max f(x, y) = x^2 - y^2 - x^3 \quad \text{when} \quad x^2 + y^2 \leq 1$$

Question 3.
A linear system of differential equations is given by

$$\begin{aligned}\dot{x} &= 2x + z - 2\\ \dot{y} &= -y - 5\\ \dot{z} &= 2y + z - 4\end{aligned}$$

a) Find the steady state $(\bar{x}, \bar{y}, \bar{z})$ of the system.

b) Rewrite the system in the form $\mathbf{w}' = A\mathbf{w}$, determine all eigenvalues of A, and find three linearly independent eigenvectors for A.

c) Find the general solution to the linear system of differential equations, and show that

$$\lim_{t\to\infty} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

when $x(0) = -1$, $y(0) = 10$, and $z(0) = -1$.

Question 4.
Let $f_1, \ldots, f_m$ be real valued functions defined on a convex set S in $\mathbb{R}^n$. Show that if $f_1, \ldots, f_m$ are convex functions and $a_1, \ldots, a_m \geq 0$, then the function

$$F = \sum_{i=1}^{m} a_i f_i$$

is convex.

Question 5.

Consider the following optimal control problem with control region $U = [0,1]$:

$$\max_{u_t \in U} \sum_{t=0}^{3} (5 - u_t) x_t^2 \quad \text{when} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = u_t x_t \\ t = 0, 1, 2, 3 \end{cases}$$

Compute the value functions $J_t(x)$ and the corresponding control functions $u_t^*(x)$ for $t = 0, 1, 2, 3$, and solve the problem when x_0 is given.

Question 6.

Let $\mathbf{a} = (a_1, a_2) \neq (0,0)$ and $\mathbf{p} = (p_1, p_2)$ be given vectors in $\mathbb{R}^2$, and consider the following Lagrange problem:

$$\min f(x_1, x_2) = (p_1 - x_1)^2 + (p_2 - x_2)^2 \quad \text{when} \quad a_1 x_1 + a_2 x_2 = 0$$

a) Find the minimum point $\mathbf{x}^* = (x_1^*, x_2^*)$ of the Lagrange problem.

b) Give a geometrical interpretation of $\mathbf{x}^*$, and show that $\mathbf{x}^*$ and $\mathbf{p} - \mathbf{x}^*$ are orthogonal vectors.