Solutions	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 14th, 2022 at 0900 - 1200

Question 1.

a) The subset $D \subseteq \mathbb{R}^n$ is clearly closed since it is defined by closed inequalities, and it is bounded since $-1 \leq x, y \leq 1$, $-3 \leq z \leq 3$, $-2 \leq w \leq 2$ for all $(x, y, z, w) \in D$, therefore D is compact. The set $D_1 = \{(x, y, z, w) : x^2 + y^2 \leq 1\}$ is convex since $x^2 + y^2 \leq 1$ defines a convex set in the plane (bounded by a circle), and the set $D_2 = \{(x, y, z, w) : 4z^2 + 9w^2 \leq 36\}$ is convex since $4z^2 + 9w^2 \leq 36$ defines a convex set in the plane (bounded by an ellipse). It follows that the intersection $D = D_1 \cap D_2$ is a convex set.

b) The Kuhn-Tucker problem is already in standard form, so we form the Lagrangian

$$\mathcal{L} = xw - yz - \lambda_1(x^2 + y^2 - 1) - \lambda_2(4z^2 + 9w^2 - 36)$$

The first order conditions (FOC) are

$$\mathcal{L}'_x = w - 2\lambda_1 x = 0, \quad \mathcal{L}'_y = -z - 2\lambda_1 y = 0, \quad \mathcal{L}'_z = -y - 8\lambda_2 z = 0, \quad \mathcal{L}'_w = x - 18\lambda_2 w = 0$$

the constraints (C) are given by $x^2 + y^2 \leq 1$ and $4z^2 + 9w^2 \leq 36$, and the complementary slackness conditions (CSC) are given by

$$\lambda_1 \geq 0 \quad \text{and} \quad \lambda_1(x^2 + y^2 - 1) = 0$$
$$\lambda_2 \geq 0 \quad \text{and} \quad \lambda_2(4z^2 + 9w^2 - 36) = 0$$

By the FOC's, $x = 0$ when $w = 0$, and $z = -2\lambda_1 y$ gives $y = -8\lambda_2(-2\lambda_1 y) = 16\lambda_1\lambda_2 y$. Hence $y(1 - 16\lambda_1\lambda_2) = 0$, which gives $y = 0$ or $\lambda_1\lambda_2 = 1/16$. If $y = 0$, then $z = 0$, and we obtain the solution

$$(x, y, z, w; \lambda_1, \lambda_2) = (0, 0, 0, 0; 0, 0)$$

of the Kuhn-Tucker conditions. Otherwise $\lambda_1\lambda_2 = 1/16$, and this means that $\lambda_1, \lambda_2 > 0$, hence $x^2 + y^2 = 1$ and $4z^2 + 9w^2 = 36$, which gives $y = \pm 1$ and $z = \pm 3$. Finally, we have $2\lambda_1 = -z/y > 0$, hence $\lambda_1 = 3/2$ and therefore $\lambda_2 = 1/24$ and y, z have opposite signs. Hence we obtain the solutions

$$(x, y, z, w; \lambda_1, \lambda_2) = (0, 1, -3, 0; 3/2, 1/24), (0, -1, 3, 0; 3/2, 1/24)$$

of the Kuhn-Tucker conditions. We conclude that the Kuhn-Tucker conditions have three solutions with $w = 0$.

Question 2.

a) The Hamiltonian is $H = p_0(2x - 3u - u^2) + p(x + u)$, where $p_0 = 1$ or $p_0 = 0$, and $(p_0, p) \neq (0, 0)$. The necessary conditions for optimum are $H'_u = p_0(-3 - 2u) + p = 0$ (since $U = \mathbb{R}$ has no boundary points), $-H'_x = -(2p_0 + p) = p'$, and $p(2) = 0$.

b) The sufficient condition in Mangasarian's criterion is that H is concave in (x, u), and this is clearly satisfied since $-p_0\, u^2$ is the unique term that is not linear in (x, u).

c) When $p_0 = 0$, the condition $p_0(-3-2u)+p = 0$ gives $p = 0$, which is not admissible since it would give $(p_0, p) = (0, 0)$. We therefore assume that $p_0 = 1$. The condition $-(2p_0 + p) = p'$ gives the differential equation $p' + p = -2$, which we solve using integrating factor:

$$(pe^t)' = -2e^t \quad \Rightarrow \quad pe^t = \int -2e^t \mathrm{d}t = -2e^t + C \quad \Rightarrow \quad p = -2 + Ce^{-t}$$

The initial condition $p(2) = 0$ then gives $0 = -2 + Ce^{-2}$, or $C = 2e^2$, and $p(t) = -2 + 2e^{2-t}$. When we substitute p in the condition $p_0(-3-2u)+p = 0$, we obtain the equation $2u = -3+p$, or

$$u = (-3+p)/2 = (-5+2e^{2-t})/2 = -5/2 + e^{2-t}$$

To find x, we use the differential equation $x' = x+u$, which can be written $x'-x = -5/2+e^{2-t}$, and multiplying with an integrating factor gives

$$(xe^{-t})' = -5/2e^{-t} + e^{2-2t} \quad \Rightarrow \quad xe^{-t} = \int -5/2e^{-t} + e^{2-2t}\mathrm{d}t = 5/2e^{-t} - 1/2e^{2-2t} + C$$

Therefore $x = 5/2 - 1/2e^{2-t} + Ce^t$. The initial condition $x(0) = 5$ gives $5 = 5/2 - 1/2e^2 + C$, or $C = 5/2 + e^2/2$. Since we obtain a normal solution (with $p_0 = 1$) and Mangasarian's necessary condition is satisfied, the optimal solution is given by

$$x(t) = \frac{5}{2} - \frac{1}{2}e^{2-t} + \frac{5}{2}e^t + \frac{1}{2}e^{2+t}$$

Question 3.

a) We have that $\mathrm{rk}(A - \lambda I) = 1$ when $\lambda = t - t^2$ since $t - \lambda = t - (t - t^2) = t^2$ and $A - \lambda I$ has echelon form

$$A - \lambda I = \begin{pmatrix} t^2 & t^2 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2 & t^2 & t^2 \\ t^2 & t^2 & t^2 & t^2 & t^2 \end{pmatrix} \to \begin{pmatrix} t^2 & t^2 & t^2 & t^2 & t^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence it has rank one when $t \neq 0$. This means that $\lambda = t - t^2$ is an eigenvalue of multiplicity $m \geq \dim E_\lambda = 4$. In fact, $m = 4$ since A is symmetric. Alternatively, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = t - t^2$ (since $m \geq 4$) and λ_5 is determined by

$$\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_5 \quad \Rightarrow \quad 5t = 4(t - t^2) + \lambda_5 \quad \Rightarrow \quad \lambda_5 = t + 4t^2$$

Since $\lambda_5 = t + 4t^2 \neq t - t^2$, the multiplicity $m = 4$.

b) We have $\det(A) = \lambda_1 \cdots \lambda_5 = (t - t^2)^4(t + 4t^2) = t^5(1-t)^4(1+4t)$.

Question 4.

a) The set V of 2×2 matrices has the natural operations addition and scalar multiplication. We have a natural map $\phi : V \to \mathbb{R}^4$ given by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b, c, d)$$

We notice that this map is a bijection, and the above mentioned operations on V correspond to addition and scalar multiplication in the Euclidean space $\mathbb{R}^4$. Therefore, V is a vector space and $\dim V = \dim \mathbb{R}^4 = 4$. Alternatively, we could also show that the operations on V satisfy the axioms of a vector space, and find a base of V to compute the dimension.

b) For the matrix A, we have that $A^T \cdot A$ is given by

$$A^T A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1^2 + 3^2 & 1 \cdot 4 + 3 \cdot 2 \\ 1 \cdot 4 + 3 \cdot 2 & 4^2 + 2^2 \end{pmatrix}$$

and therefore that $||A||^2$ is given by

$$||A||^2 = \mathrm{tr}(A^T A) = 1^2 + 3^2 + 4^2 + 2^2 = 30$$

and $||A|| = \sqrt{30}$.

c) For a general matrix A in V, we have that $A^T A$ is given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & d^2 + b^2 \end{pmatrix}$$

This means that $||A||^2$ is given by

$$||A||^2 = \mathrm{tr}(A^T A) = a^2 + c^2 + d^2 + b^2$$

and therefore $||A|| = \sqrt{a^2 + b^2 + c^2 + d^2}$, which corresponds to the Euclidean norm on $\mathbb{R}^4$ via ϕ. It follows that $||A||$ is a norm on V. Alternatively, we can check that the axioms for a norm is satisfied for $||A|| = \sqrt{\mathrm{tr}(A^T A)}$.