Exam	Final exam in DRE 7017 Mathematics, Ph.D.
Date	October 14th, 2022 at 0900 - 1200

This exam consists of 10 problems of equal weight. **You must give reasons for your answers.**

Question 1.
We consider the Kuhn-Tucker problem

$$\max\ f(x,y,z,w) = xw - yz \text{ subject to } \begin{cases} x^2 + y^2 \le 1 \\ 4z^2 + 9w^2 \le 36 \end{cases}$$

a) Show that the set D of points satisfying both constraints is compact. Is it convex?

b) Write down the Kuhn-Tucker conditions, and find all solutions with $w = 0$.

Question 2.
Consider the optimal control problem

$$\max \int_0^2 (2x - 3u - u^2)\,\mathrm{d}t \text{ when } \begin{cases} x(0) = 5 \\ x' = x + u \\ u \in U = \mathbb{R} \end{cases}$$

a) Write down the Hamiltonian and the necessary conditions for optimum.

b) Show that the sufficient condition in Mangasarian's criterion is satisfied.

c) Solve the problem.

Question 3.
Let t be a parameter with $t \neq 0$, and consider the matrix

$$A = \begin{pmatrix} t & t^2 & t^2 & t^2 & t^2 \\ t^2 & t & t^2 & t^2 & t^2 \\ t^2 & t^2 & t & t^2 & t^2 \\ t^2 & t^2 & t^2 & t & t^2 \\ t^2 & t^2 & t^2 & t^2 & t \end{pmatrix}$$

a) Show that $\lambda = t - t^2$ is an eigenvalue of A, and determine its multiplicity.

b) Use the result in a) to compute the determinant of A.

Question 4.
Let V be the set of all 2×2 matrices.

a) Explain that V is a vector space, and find its dimension.

b) For any matrix $A \in V$, we define $||A|| = \sqrt{\mathrm{tr}(A^T A)}$. Compute $||A||$ when A is the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

c) Prove that $||A|| = \sqrt{\mathrm{tr}(A^T A)}$ is a norm on V.