Exam	Exercise Problems in DRE 7017 Mathematics, Ph.D.
Date	September 2022

Question 1.
We consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) Find a base of $\mathrm{Null}(A - I)$ and $\mathrm{Null}((A - I)^2)$. Which of the vectors are eigenvectors of A?
(b) Determine whether A is diagonalizable.
(c) A vector $\mathbf{v}$ is called a *generalized eigenvector* for A if $(A - \lambda I)^n \cdot \mathbf{v} = \mathbf{0}$ for some real number λ and some integer $n \geq 1$. Explain that any eigenvector for A is a generalized eigenvector, and find 3 generalized eigenvectors of A that are linearly independent.
(d) Determine the definiteness of the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.
(e) Solve the optimization problem $\min q(\mathbf{x})$ when $x + y + z = 1$.

Question 2.
Let $V = \mathrm{C}(I, \mathbb{R})$ be the vector space of continuous functions on the unit interval $I = [0, 1]$, equipped with the sup-norm, and define the function $f_n(x) = \sqrt[n]{x}$ for all positive integers $n \geq 1$.

(a) Compute $d(f_2, f_3)$ and $d(f_3, f_4)$.
(b) Show that if (f_n) converges to f in V, then $f_n(x)$ converges to $f(x)$ for all $x \in I$.
(c) Is (f_n) a Cauchy sequence?
(d) Does (f_n) have a convergent subsequence?

Question 3.
Solve the optimal control problem

$$\max \sum_{t=0}^{5} \left(1 + 3x_t^2 - 2u_t^2\right) \quad \text{when} \quad \begin{cases} x_0 = 4 \\ x_{t+1} = x_t + u_t \\ u_t \in U \end{cases}$$

with control region $U = [0, 1]$.

Question 4.
Solve the optimal control problem

$$\max \int_0^2 10 - (y - u)^2 e^{-t} \mathrm{d}t \quad \text{when} \quad \begin{cases} y' = y - u/2 \\ y(0) = 0 \\ y(2) = 6e \end{cases}$$