Problem Set I: Solutions

DRE 7017

1. $A \cdot \underline{v} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 5 & a & 11 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for all a $\Rightarrow$ $\underline{v} \in \text{Null}(A)$ for all a.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 5 & a & 11 \end{pmatrix} \begin{matrix} \\ -3 \\ -2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 3 & a-4 & 9 \end{pmatrix} \begin{matrix} \\ \\ 3 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & a-19 & 0 \end{pmatrix}$$

$a \neq 19$: $\dim \text{Null}(A) = 1$ Base: $B = \{\underline{v}\}$

$a = 19$: $\dim \text{Null}(A) = 2$ Base: $B = \{\underline{u}, \underline{v}\}$, where $\underline{u}, \underline{v}$ are given by:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x + y + 2z + w &= 0 \\ -y - 5z - 3w &= 0 \end{aligned}$$

$\Downarrow$

$y = -5z - 3w$

$x = -y - 2z - w = 5z + 3w - 2z - w = 3z + 2w$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \cdot \underbrace{\begin{pmatrix} 3 \\ -5 \\ 1 \\ 0 \end{pmatrix}}_{\underline{u}} + w \cdot \underbrace{\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}}_{\underline{v}}$$

2. For a), b) see earlier solution for eigenvalues/eigenvectors

A is not orthogonormal diagonalizable since A is not symmetric.

c) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)^2 \cdot (4-\lambda) = 0$$

$\lambda_1 = \lambda_2 = 1, \ \lambda_3 = 4$

Since $\lambda = 1$ gives $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ a matrix of rank 1, hence $\lambda = 1$ has at least mult. $2 = 3 - 1$, and $\lambda_1 = \lambda_2 = 1$ gives $\text{tr}(A) = 2 + 2 + 2 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 6 - 2 = 4$

Eigenvectors:

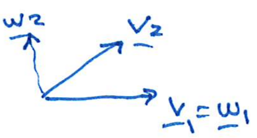
$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{matrix} x + y + z = 0 \\ y, z \text{ free} \end{matrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \cdot \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\underline{v_1}} + z \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\underline{v_2}}$$

$E_1 = \text{span}(\underline{v_1}, \underline{v_2})$

Orthogonal base:

$\underline{w_1} = \underline{v_1}$, $\underline{w_2} = \underline{v_2} - \text{proj}_{\underline{v_1}} \underline{v_2} = \underline{v_2} - \dfrac{\langle \underline{v_2} \cdot \underline{v_1} \rangle}{\langle \underline{v_1}, \underline{v_1} \rangle} \underline{v_1}$



$= (-1, 0, 1) - \frac{1}{2}(-1, 1, 0)$

$= (-\frac{1}{2}, -\frac{1}{2}, 1) = \frac{1}{2}(-1, -1, 2)$

orthonormal base of E_1:

$\frac{1}{\|w_1\|}\cdot w_1 = \frac{1}{\sqrt{2}}(-1,1,0)$, $\frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{6}}(-1,-1,2)$

check: ok

$w_1' \cdot w_2' = 0$

$w_1' \cdot w_1' = 1$

$w_2' \cdot w_2' = 1$

$\lambda=4$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$-x - y + 2z = 0$

$-3y + 3z = 0$

$y = z$, $x = -y + 2z = z$

$\Rightarrow (x,y,z) = (z,z,z)$

$= z\cdot(1,1,1)$

$v_3 = (1,1,1)$

orthonormal base of E_4:

$\frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{3}}(1,1,1)$

$\Downarrow$

Orthonormal base of $\mathbb{R}^3$ of eigenvectors of A:

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0),\ \frac{1}{\sqrt{6}}(-1,-1,2),\ \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

3. a) c) see earlier solution

b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & a \end{pmatrix}$ $D_1 = 1$, $D_2 = 0$, $D_3 = 0$

Row 2 = 2·Row 1 $\Rightarrow$ rk $A = 2$, $|A| = 0$

$\Delta_1 = 1, 4, a$

$\Delta_2 = 0, 4a-4, a-1$

$\Delta_3 = 0$

(not pos. defin. since $D_2 = 0$)

$a \geq 1$: $\Delta_1, \Delta_2, \Delta_3 \geq 0 \Rightarrow$ positive semidefin.

$a < 1$: $\Delta_2^{13} = a - 1 < 0 \Rightarrow$ indefinite

4. See earlier solution.