EBA 2911, lecture 10, 16 Oct 2019, Runar Ile

1. Inverse functions
2. Exponential functions
3. Logarithms

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1. Inverse functions

- table of function values
- expression
- graph

<u>Ex</u>: $f(x) = (x-3)^2$ with domain $D_f = [3, \rightarrow\rangle$ $(x \geq 3)$

<u>Table of function values:</u>

x	3	4	5	6	7	$g(x)$
$f(x)$	0	1	4	9	16	x

← the inverse function

so $g(0) = 3$, $g(1) = 4$, $g(4) = 5$ ....

then

$f(g(0)) = f(3) = 0$ $\qquad$ $g(f(3)) = g(0) = 3$

$f(g(1)) = f(4) = 1$ $\qquad$ and $\qquad$ $g(f(4)) = g(1) = 4$

$f(g(4)) = f(5) = 4$ $\qquad$ $g(f(5)) = g(4) = 5$

Definition: $f(x)$ with domain D_f and
$g(x)$ with domain D_g
are inverse functions if
$f(g(x)) = x$ for all x in D_g and $g(f(x)) = x$ for all x in D_f

Problem: Show that $f(x) = \sqrt{x-1}$ with $D_f = [1, \to\rangle$ $(x \geq 1)$
and $g(x) = x^2+1$ with $D_g = [0, \to\rangle$ $(x \geq 0)$
are inverse functions.

Solution: Note that $g(-3) = (-3)^2 + 1 = 9+1 = 10$
and $f(10) = \sqrt{10-1} = \sqrt{9} = 3$ and so
$f(g(-3)) = 3$ (and not -3) - not good....?
- but -3 is not in D_g, so no problem here.

For all numbers x we have $f(g(x)) = \sqrt{g(x)-1}$
$= \sqrt{(x^2+1)-1} = \sqrt{x^2} = |x|$ and for
$x \geq 0$ $|x| = x$ so ok.

For $x \geq 1$, $g(f(x)) = (f(x))^2 + 1 = (\sqrt{x-1})^2 + 1$
$= x - 1 + 1 = x$ so ok.

How to find an expression for the inverse function?

① Solve the equation $y = f(x)$ for x

② Switch the variables x and y.

③ Put $D_g = R_f$ (the range of f)

<u>Ex</u>: $f(x) = (x-3)^2$ with $D_f = [3, \rightarrow>$

① Solve the equation $y = (x-3)^2$ for x

– take the square root on each side

$$\sqrt{y} = \sqrt{(x-3)^2} \qquad (y \geq 0)$$

that is $\sqrt{y} = |x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ -x+3 & \text{if } x \leq 3 \end{cases}$

So $\sqrt{y} = x - 3$ for x in $D_f = [3, \rightarrow>$

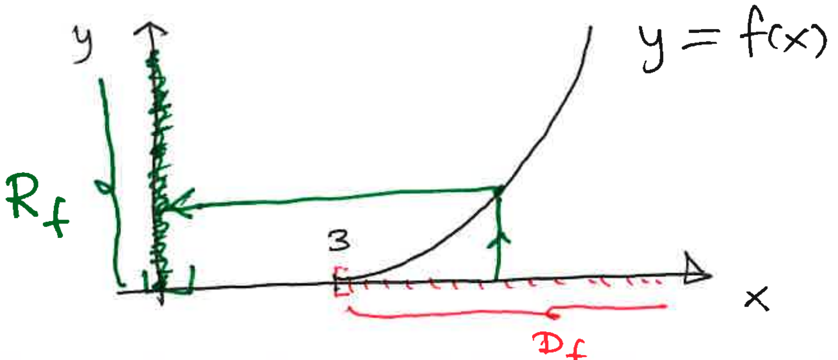
and then $x = 3 + \sqrt{y}$

② Switch variables: $y = g(x) = 3 + \sqrt{x}$

③ $D_g = R_f = [0, \rightarrow>$ because

$f(x) = (x-3)^2 = y$ has a

solution for $x \geq 3$ for all $y \geq 0$.



Problem: Let $f(x) = (x-3)^2$ with $D_f = \langle \leftarrow, 3]$ $(x \leq 3)$

Determine the expression for the inverse function $g(x)$ and its domain.

Solution:

① solve the eq $y = (x-3)^2$

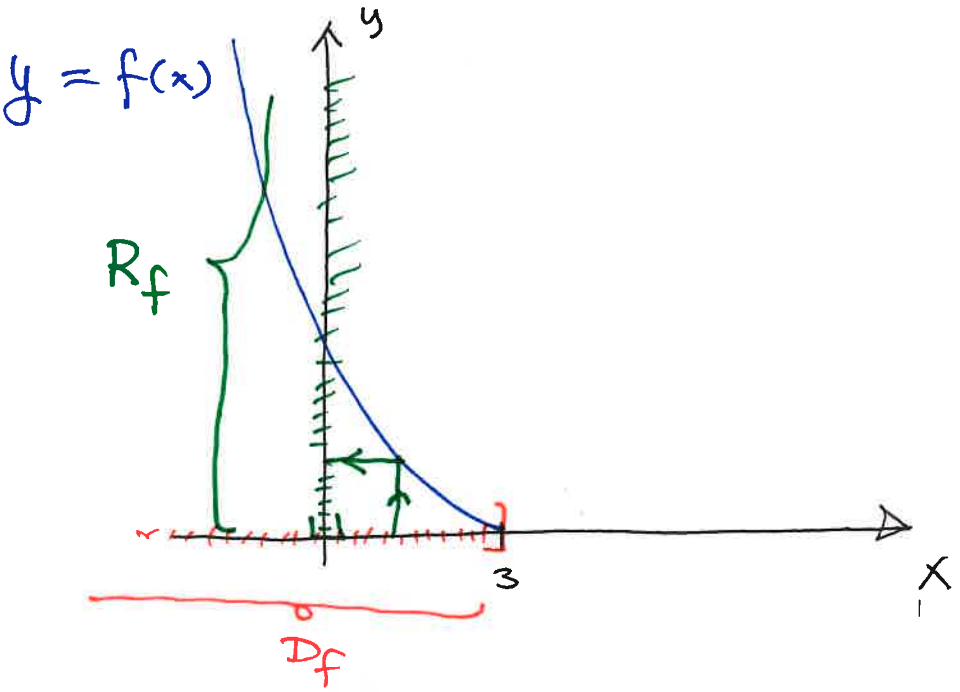
$$\sqrt{y} = |x-3| = \begin{cases} x-3 & \text{for } x \geq 3 \\ -x+3 & \text{for } x < 3 \end{cases}$$

so $\sqrt{y} = -x+3$ for $x \in D_f = \langle \leftarrow, 3]$

and then $x = 3 - \sqrt{y}$

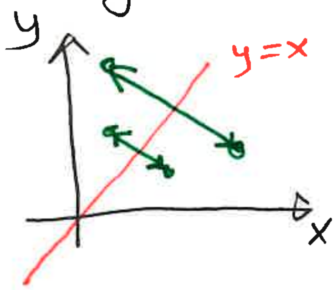
② Switch variables: $y = g(x) = 3 - \sqrt{x}$

③ $D_g = R_f = [0, \rightarrow\rangle$ because $f(x) = (x-3)^2 = y$ has a solution for x with $x \leq 3$ for all $y \geq 0$.



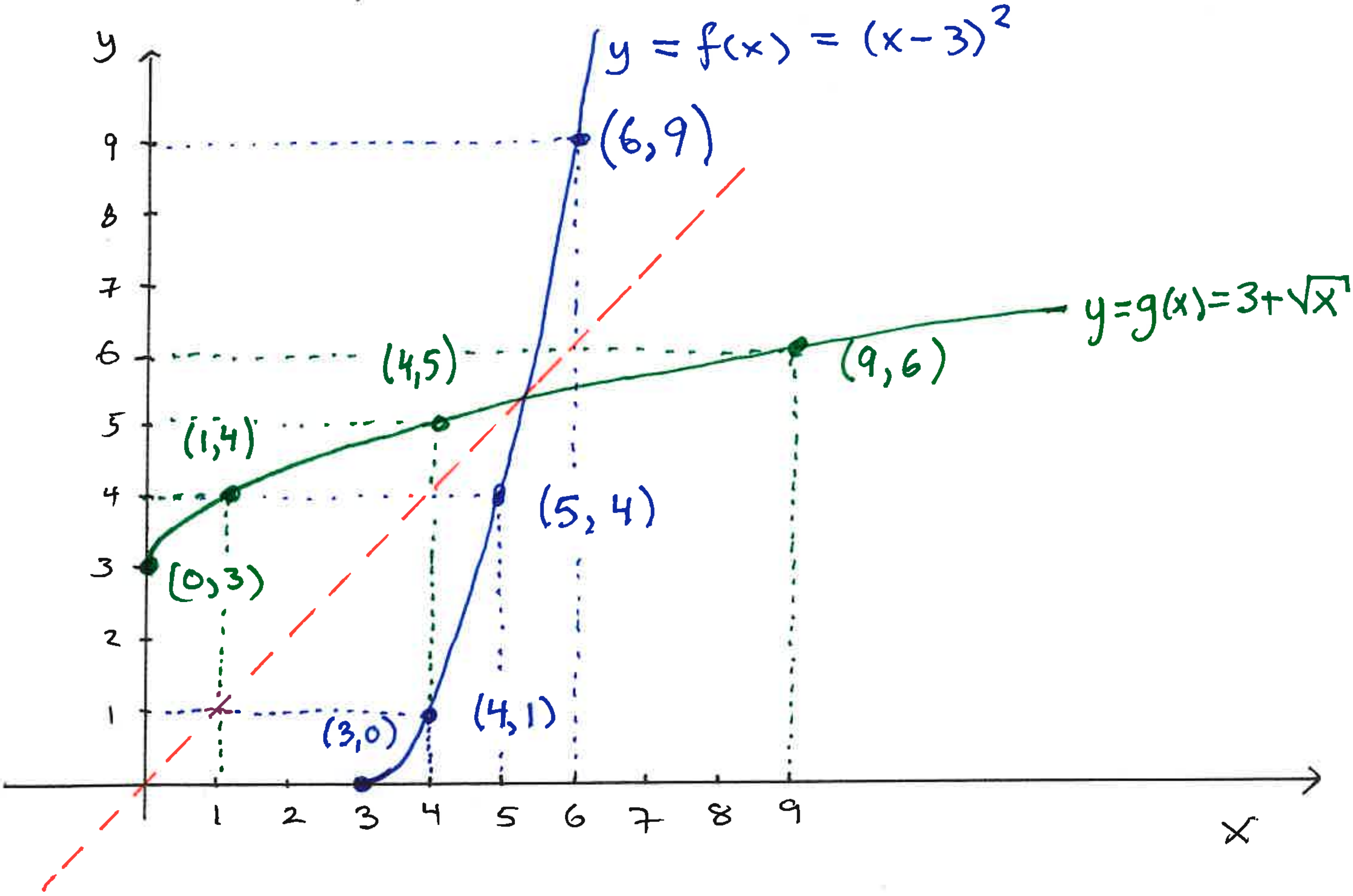
The graph of the inverse function

- is the mirror image of the graph of $f(x)$ with respect to the 'diagonal' $y = x$



Ex: $f(x) = (x-3)^2$ with $D_f = [3, \rightarrow\rangle$

x	3	4	5	6	7	$g(x)$
$f(x)$	0	1	4	9	16	x

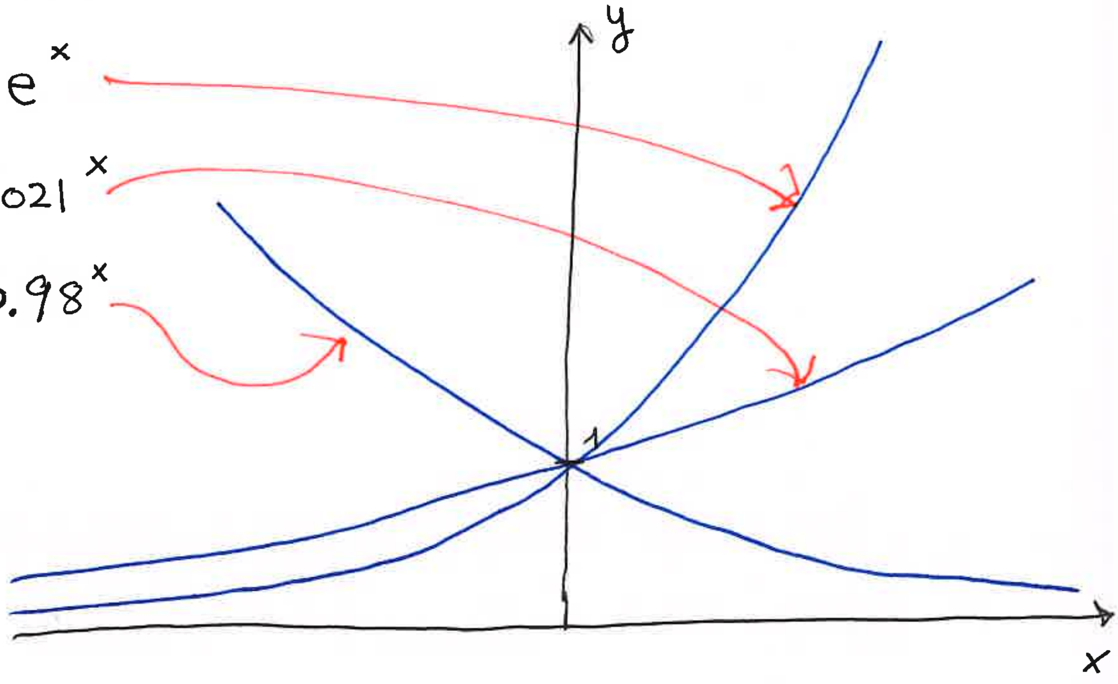


2. Exponential functions

Ex: $f(x) = e^x$

$f(x) = 1.021^x$

$f(x) = 0.98^x$



$\underline{a > 1}$: $f(x) = a^x$ is strictly increasing without bounds

and $a^x \xrightarrow[x \to -\infty]{} 0^+$

(say $a^{-100} = \dfrac{1}{a^{100}}$ very close to 0)

$\underline{0 < a < 1}$: $f(x) = a^x$ is strictly decreasing

and $a^x \xrightarrow[x \to \infty]{} 0^+$

(Note: a is always positive)

D_f = all numbers and $R_f = \langle 0, \rightarrow \rangle$

Problem: Kåre deposits 4000 into an account earning 1.2 % interest. Determine the expression $f(x)$ which gives the balance after x years with

a) annual compounding

b) continuous compounding

Answers: a) $f(x) = 4000 \cdot 1.012^x$

b) $f(x) = 4000 \cdot (e^{0.012})^x$

$= 4000 \cdot e^{0.012 \cdot x}$

Power rules: If $f(x) = a^x$ then

$$f(x) \cdot f(y) = a^x \cdot a^y = a^{x+y} = f(x+y)$$

and $\frac{1}{f(x)} = \frac{1}{a^x} = a^{-x} = f(-x)$

3. Logarithms Suppose $a > 0$ and $a \neq 1$

Then $g(x) = \log_a(x)$ is the inverse function of $f(x) = a^x$ and $D_g = R_f = \langle 0, \rightarrow \rangle$

(a is called the base of the logarithm)

Ex: $a = 2$, $\log_2(10)$ = the number which 2 has to be raised to to give 10.

$2^{3.322} \approx 10$

so $\log_2(10) \approx 3.322$

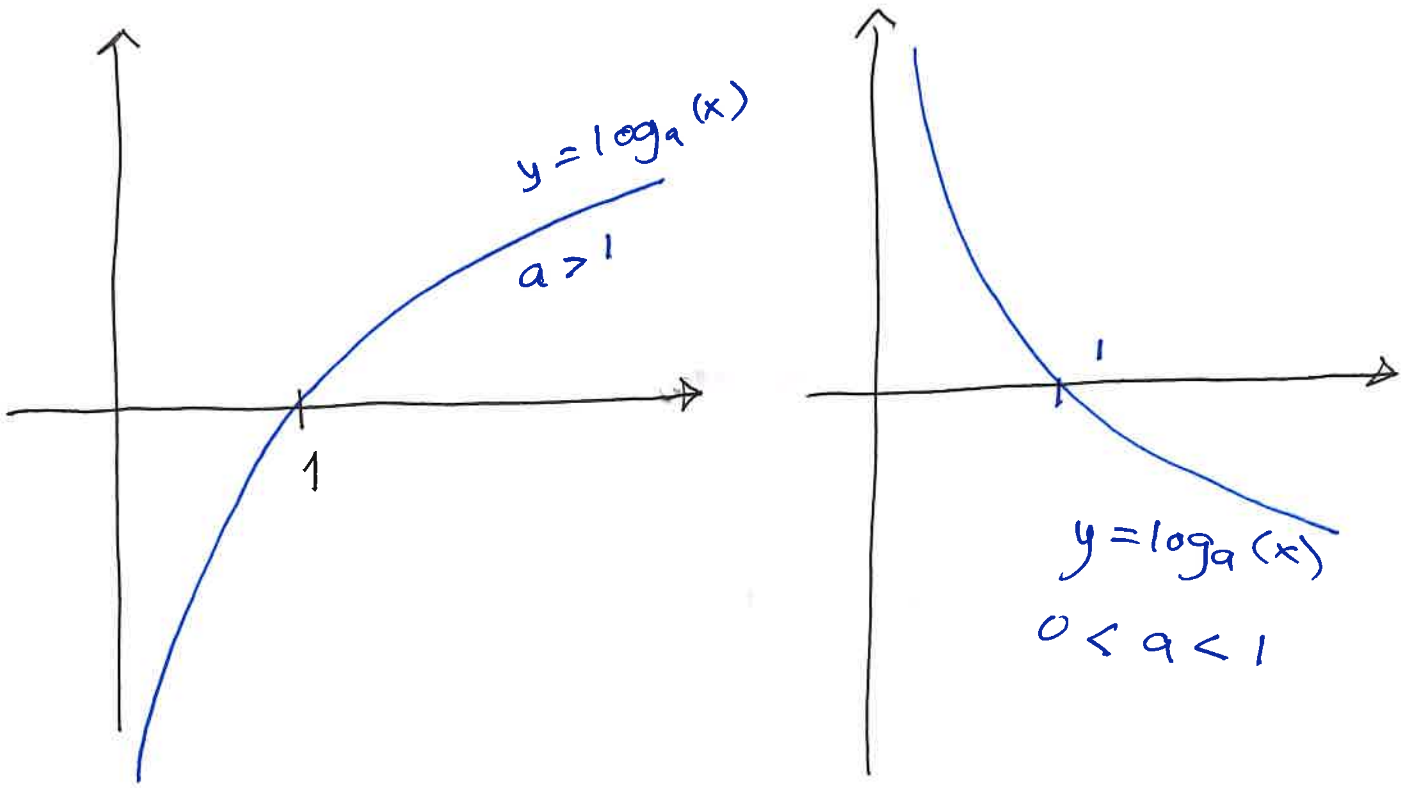
x	1	3	3.322	4	-1	0	$\log_2(x)$
2^x	2	8	10	16	0,5	1	x

Problems: Compute $\log_2(16)$ and $\log_2(0.5)$ and $\log_2(1)$

Solutions: Since $2^4 = 16$ $\quad$ since $2^{-1} = 0,5$ $\quad$ since $2^0 = 1$

$\log_2(16) = 4$ $\quad$ $\log_2(0,5) = -1$ $\quad$ $\log_2(1) = 0$

Graphs:



Rules:

① $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

② $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

③ $\log_a(x^r) = r \cdot \log_a(x)$

Definition: $\ln(x) = \log_e(x)$ $\quad$ e = Euler number

- is called the natural logarithm.