EBA 2911, Lecture 11, 23 Oct. 2019, Runar Ile

1. Repetition & problems
2. Tangents and the derivative
3. The derivative as a function
4. Rules of differentiation

1. Rep. & prob.

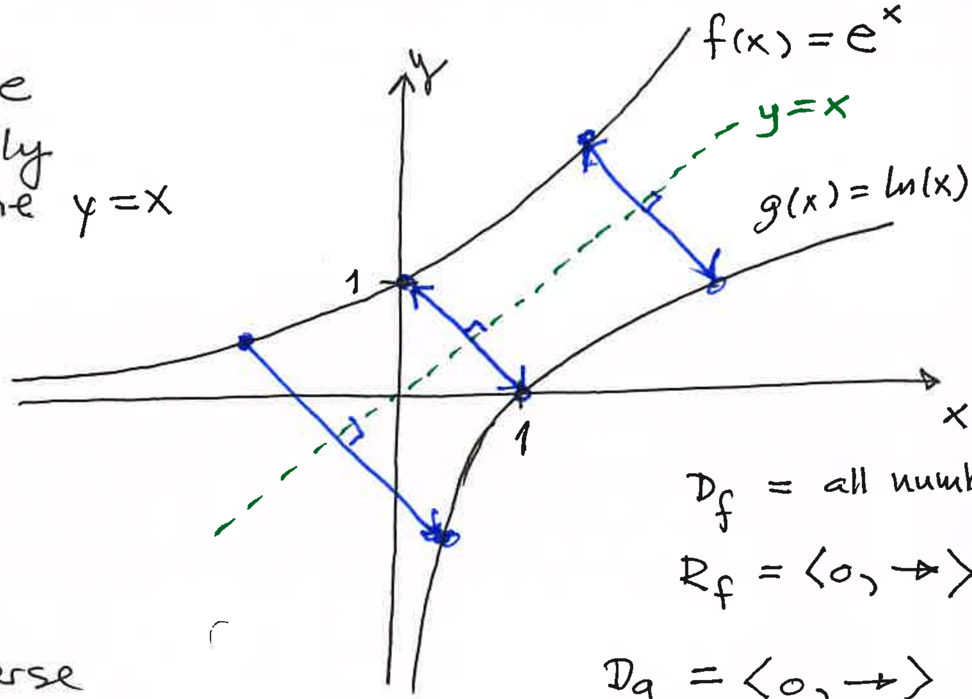
Inverse functions

Definition

$f(g(x)) = x$ for all x in D_g

$g(f(x)) = x$ for all x in D_f

*) The graphs lie symmetrically about the line $y=x$



D_f = all numbers

$R_f = \langle 0, \rightarrow \rangle$

$D_g = \langle 0, \rightarrow \rangle$

R_g = all numbers

*) For $f(x)$ to have an inverse function, $f(x)$ has to be either strictly increasing or strictly decreasing

*) Note: $D_g = R_f$ and $R_g = D_f$

How to find the expression for the inverse function?

Prob. 2d $\quad f(x) = 20 + \frac{1}{x-3}$, $D_f = \langle 3, \rightarrow \rangle$
$(x > 3)$

we find the inverse function $g(x)$ with D_g.

① we solve the equation $y = f(x)$ for x.

That is: $$y = 20 + \frac{1}{x-3}$$

multiply with $(x-3)$ on both sides

$$y \cdot (x-3) = 20 \cdot (x-3) + \frac{1}{\cancel{x-3}} \cdot \cancel{(x-3)}$$

resolve parentheses

$$yx - 3y = 20x - 60 + 1$$

collect all terms with x on the left h.s. and the rest on the right h.s.

$$yx - 20x = 3y - 59$$

factorise x out on the left h.s.

$$(y-20)x = 3y - 59$$

divide by $(y-20)$ on each side

$$x = \frac{3y-59}{y-20}$$ (assume $y \neq 20$)

polynomial division

$$= 3 + \frac{1}{y-20}$$

② switch the variables x and y.

$$y = g(x) = 3 + \frac{1}{x-20}$$

③ Put $D_g = R_f$ and determine it.

R_f = the set of y-values 'hit' by $f(x)$ for x in D_f, that is those y such that

$$y = \underbrace{20 + \frac{1}{x-3}}_{f(x)}$$ has a solution x in $D_f = \langle 3, \rightarrow \rangle$

But when $x > 3$, $\frac{1}{x-3}$ is positive, so $20 + \frac{1}{x-3}$ is larger than 20.

In ① we saw that all $y > 20$ were in the range of $f(x)$, so

$$R_f = \langle 20, \rightarrow \rangle = D_g .$$

Exponential - and logarithmic functions

$a > 0$ $\quad f(x) = a^x$, D_f = all numbers

$a > 0$ and $a \neq 1$ $\quad g(x) = \log_a(x)$, $D_g = \langle 0, \rightarrow \rangle = R_f$

Ex: How much time is needed for the deposit in an account earning 3% interest to double?

Solution: We solve the equation $1.03^x = 2$ (*) which gives $x = \log_{1.03}(2)$ by definition!

- but we cannot calculate $\log_{1.03}(x)$ directly on the calculator!

Instead we insert the left h.s. and the right h.s. of (*) into $\ln(x) = \log_e(x)$:

$$\ln(1.03^x) = \ln(2)$$

$$x \cdot \ln(1.03) = \ln(2)$$

$$x = \frac{\ln(2)}{\ln(1.03)} = 23.45$$

This also gives $\log_{1.03}(2) = \frac{\ln(2)}{\ln(1.03)}$

<u>Prob. 6c</u> $\frac{3e^x}{e^x+1} < 5$

- we could put $u = e^x$ to get $\frac{3u}{u+1} < 5$ and solve for u and the use $u = e^x$

But since $e^x + 1 > 0$ for all x, we can multiply each side by $e^x + 1$ and keep the inequality sign:

$$3e^x < 5(e^x+1) = 5e^x + 5$$

$$3e^x - 5e^x < 5$$

$$-2e^x < 5 \quad \text{divide by } -2 \text{ on each}$$

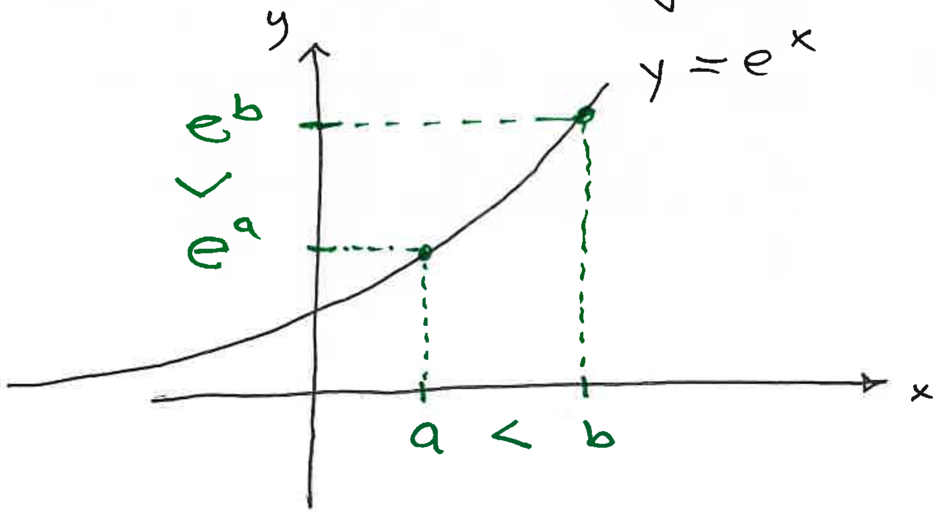
$$e^x > -\frac{5}{2}$$

which is true for all numbers x.

Prob 6b

$\ln(x-3) < -2 \qquad (x > 3)$

Because e^x is strictly increasing for all x we can insert the left h.s. and the right h.s. into e^x and keep the inequality:



$$e^{\ln(x-3)} < e^{-2}$$

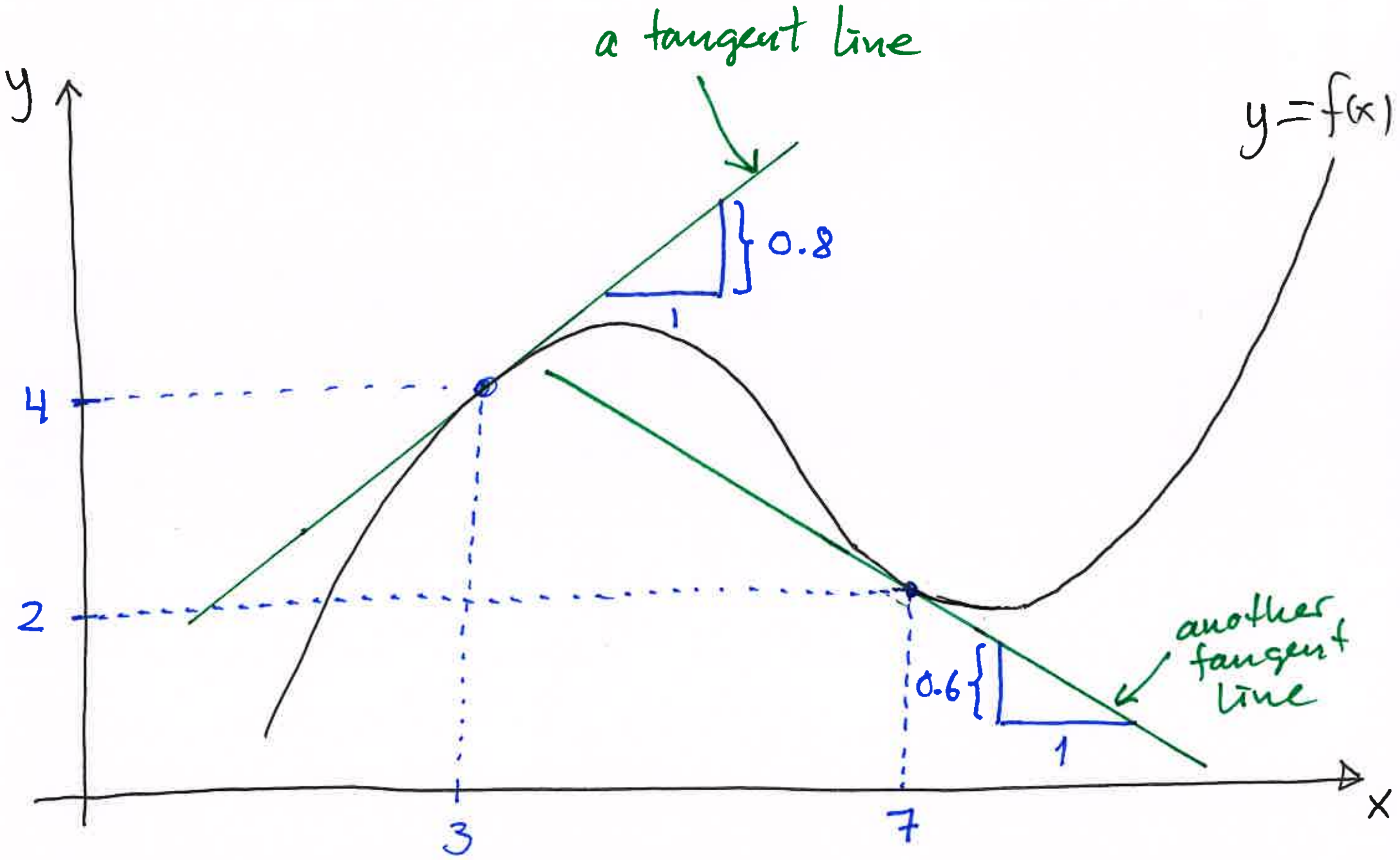
$$x - 3 < e^{-2}$$

so $\quad x < 3 + e^{-2}$

so $\quad \underline{\underline{x \in \langle 3, 3+e^{-2} \rangle}}$

or $\quad \underline{3 < x < 3 + e^{-2}}$

2. Tangents and the derivative



The tangent of the graph of $f(x)$ at the point $(3, 4)$ has slope 0.8

We write $f'(3) = 0.8$

The tangent of the graph of $f(x)$ at the point $(7, 2)$ has slope -0.6

We write $f'(7) = -0.6$

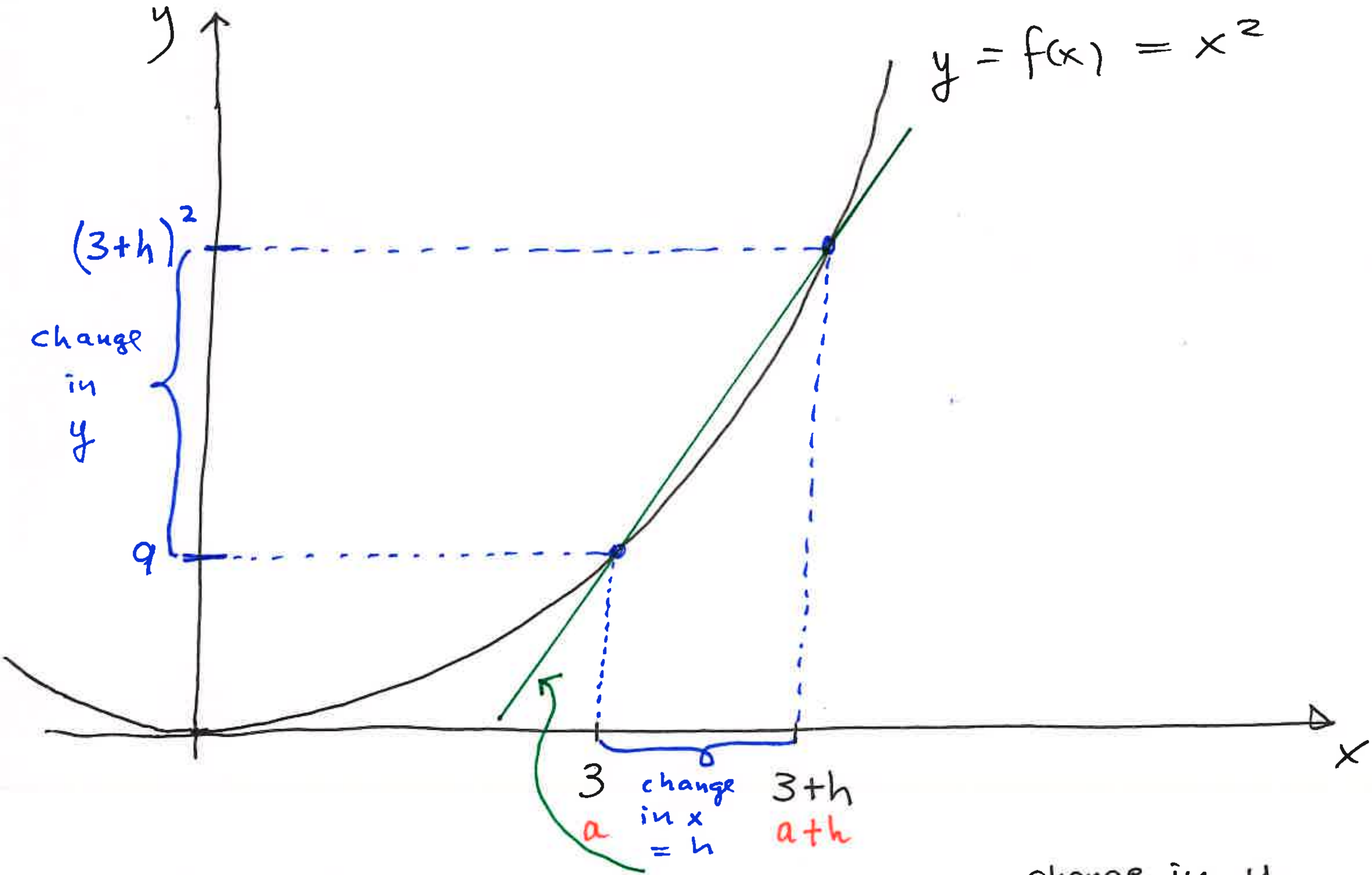
Two important applications:

1) To determine where the function increases/decreases and where it has maximum and minimum.

2) Approximate complicated functions with linear functions.
 - typical in economic models

How to find the slope of the tangent.

Ex: $f(x) = x^2$. What is the slope of the tangent through $(3, 9)$



The slope of the <u>secant</u> <u>line</u> $= \dfrac{\text{change in } y}{\text{change in } x}$

$$= \frac{(3+h)^2 - 9}{h} = \frac{(3+h)(3+h) - 9}{h}$$

$$\frac{(a+h)^2 - a^2}{h} = \frac{(a+h)(a+h) - a^2}{h}$$

$$= \frac{\cancel{9} + 2\cdot 3\cdot h + h^2 \cancel{-9}}{h} = \frac{6h + h^2}{h} = \frac{(6+h)h}{h}$$

$$\frac{\cancel{a^2} + 2\cdot ah + h^2 \cancel{- a^2}}{h} = \frac{2\cdot ah + h^2}{h} = \frac{(2a+h)h}{h}$$

$= 6 + h \xrightarrow[h \to 0]{} 6$ which has to be the slope of the tangent to $f(x)$ in $(3, 9)$.

$2a + h \longrightarrow 2a$

We write $f'(3) = 6$

3. The derivative as a function

In ex: If we put $x = a$ instead of $x = 3$

we get $f'(a) = 2a$

– the derivative is a function!

$$f'(x) = 2x$$

Ex: The slope of the tangent of $f(x)$ at $(-3, 9)$ is

$$f'(-3) = 2\cdot(-3) = -6$$

We could do the same with $f(x) = x^3$

– would get (after more calculations) that $f'(x) = 3\cdot x^2$.

4. Rules of differentiation.

Power rule: $\boxed{f(x) = x^n \text{ gives } f'(x) = n\cdot x^{n-1} \text{ for all } n}$

Ex: $f(x) = x^{10}$, $f'(x) = 10\cdot x^9$

Ex: $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ so

$$f'(x) = \frac{1}{3}\cdot x^{-\frac{2}{3}} = \frac{1}{3\cdot\sqrt[3]{x^2}}$$

The sum (subtraction) rule: If $f(x) = g(x) + h(x)$

then $f'(x) = g'(x) + h'(x)$

Ex: $f(x) = x + x^3$ then $f'(x) = 1 + 3x^2$

The constant rule: If k is a constant number

and $f(x) = k \cdot g(x)$, then

$f'(x) = k \cdot g'(x)$

Ex: $k = 7$, $g(x) = x^2$, then $f(x) = 7x^2$

and $f'(x) = 7 \cdot 2x = 14x$

The product rule: If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Ex: $f(x) = (5x^3 - 2x + 1)(3x + 7)$

will find $f'(x)$ by using the product r.

$g(x) = 5x^3 - 2x + 1$ $\quad$ $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $\quad$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2)(3x + 7) + (5x^3 - 2x + 1) \cdot 3$

note the parenthesis!

calculate

$= 60x^3 + 105x^2 - 12x - 11$

The quotient rule: Suppose $f(x) = \frac{g(x)}{h(x)}$

Then $$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Ex: $f(x) = \frac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$
$g'(x) = 3$ and $h'(x) = 2$

note the paran.

$$f'(x) = \frac{3 \cdot (2x+5) - (3x+1) \cdot 2}{(2x+5)^2}$$

$$= \frac{3 \cdot 2x + 3 \cdot 5 - (3x \cdot 2 + 1 \cdot 2)}{(2x+5)^2}$$

$$= \frac{6x + 15 - (6x + 2)}{(2x+5)^2} = \frac{13}{(2x+5)^2}$$

usually better not to expand denominator!

The chain rule

If $f(x) = g(u(x))$

the exterior function

the kernel, or the inner function

- a function composed of two functions $g(u)$ and $u(x)$.

then $f'(x) = g'(u) \cdot u'(x)$

Ex: $f(x) = (x^2+2)^{10}$

$u = u(x) = x^2+2$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\qquad$ $g'(u) = 10 \cdot u^9$

Then $f'(x) = 10u^9 \cdot 2x$

$$= 10(x^2+2)^9 \cdot 2x$$

$$= \underline{\underline{20x(x^2+2)^9}}.$$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\qquad$ $g'(x) = \frac{1}{x}$