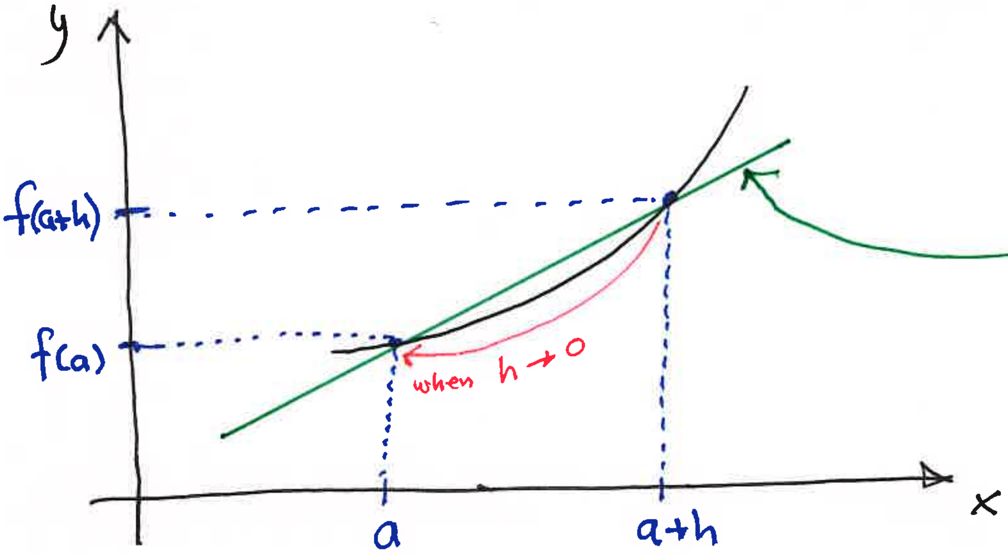
EBA 2911, lecture 12, 30 Oct. 2019, Runar Ile

1. Repetition and problems
2. Optimisation

1. Rep. & problems

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

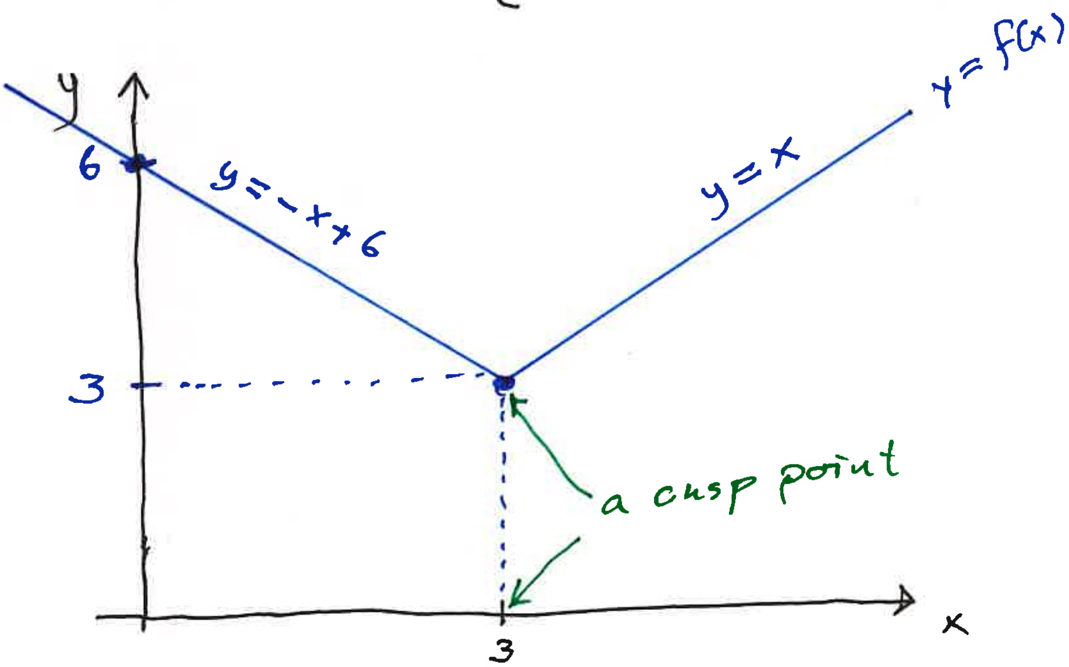
$\frac{f(a+h)-f(a)}{h}$ = the slope of the secant



Note: The derivative does not always exist!

Ex: $f(x) = |x-3| + 3 = \begin{cases} -(x-3)+3 & \text{if } x-3<0 \\ x-3+3 & \text{if } x-3 \geq 0 \end{cases}$

$$= \begin{cases} -x+6 & \text{if } x<3 \\ x & \text{if } x \geq 3 \end{cases}$$



Here $f'(x) = \begin{cases} -1 & \text{if } x<3 \\ 1 & \text{if } x>3 \end{cases}$

But for $x = 3$ there is no tangent! Hence $f'(3)$ does not exist

Difficult to use the definition of $f'(a)$
Easier to find the derivative function
$f'(x)$.

Basic results:

$$[\ln(x)]' = \frac{1}{x}$$

$$(e^x)' = e^x$$

$$(x^n)' = n \cdot x^{n-1} \quad (\text{for all } n)$$

Ex: $(x^{3.19})' = 3.19 \cdot x^{2.19}$

The product rule: $[g(x) \cdot h(x)]' = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Note: $(x^2)' = 2x$, but $(x)' = 1$ and $1 \cdot 1 = 1$, not $2x$.

$$\neq$$
$$(x)' \cdot (x)' = 1$$

Prob 26 $f(x) = 30x^2 e^x \ln(x)$

$$f'(x) \overset{\text{prod.r.}}{=} 30\left([x^2 e^x]' \cdot \ln(x) + x^2 e^x \cdot [\ln(x)]'\right)$$

$$\overset{\text{prod.r.}}{=} 30\left(\underbrace{[2x \cdot e^x + x^2 \cdot e^x]}_{e^x \text{ is a common factor}} \cdot \ln(x) + x^{\cancel{2}} e^x \cdot \frac{1}{\cancel{x}}\right)$$

$$= 30\left(\underbrace{[2x + x^2]}_{x \text{ is a common factor}} e^x \ln(x) + x e^x\right)$$

$$= 30\left(\underbrace{[2+x]\, x e^x \ln(x) + x e^x}_{x e^x \text{ is a common factor}}\right)$$

$$= 30x\left((2+x)\ln(x) + 1\right) e^x$$

The quotient rule: $f(x) = \dfrac{g(x)}{h(x)}$ then

$$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Problem 3h $f(x) = \dfrac{2\ln(x)}{3e^x}$ gives

$$f'(x) = \frac{[2\ln(x)]' \cdot 3e^x - 2\ln(x) \cdot (3e^x)'}{(3e^x)^2}$$

$$= \frac{2 \cdot \frac{1}{x} \cdot 3e^x - 2\ln(x) \cdot 3e^x}{9e^{2x}} \quad \Big| \cdot \frac{x}{x} = 1$$

$$= \frac{6e^x - 6x\ln(x)e^x}{9xe^{2x}}$$

$6e^x$ is a common factor

$$= \frac{\overset{2}{\cancel{6}}(1 - x\ln(x))\cancel{e^x}}{\underset{3}{\cancel{9}}x\,e^{\cancel{2x}}}$$

$e^{2x} = e^x \cdot e^x$

$$= \frac{2(1 - x\ln(x))}{3xe^x}$$

Chain rule: $f(x) = g(u(x))$

gives $f'(x) = g_u'(u(x)) \cdot u'(x)$

differentiation with respect to u: $g'(u)$, and then insert $u = u(x)$.

Problem 5, last one

$$f(x) = \frac{2}{(2x+1)\sqrt{2x+1}}$$

$u = u(x) = 2x+1$ and $g(u) = \frac{2}{u\sqrt{u}}$

then $u'(x) = 2$

$$= \frac{2}{u^1 \cdot u^{\frac{1}{2}}} = \frac{2}{u^{1,5}}$$

$$= 2 \cdot u^{-1,5}$$

$$g'(u) = 2 \cdot (-1,5)\, u^{-1,5-1}$$

$$= -3u^{-2.5}$$

$$= \frac{-3}{u^2\sqrt{u}}$$

Apply the chain rule:

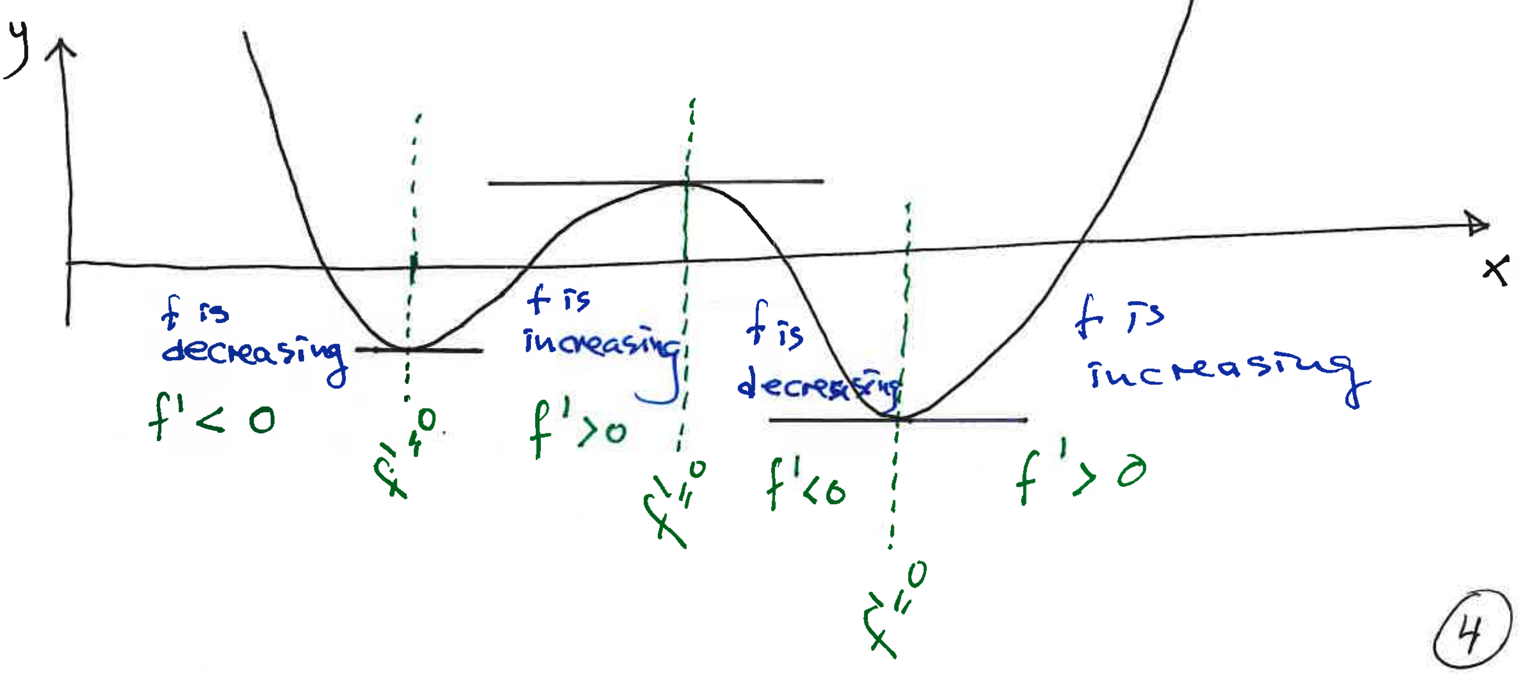
$$f'(x) = g'(u) \cdot u'(x)$$

$$= \frac{-3}{u^2\sqrt{u}} \cdot 2$$

$$= \frac{-6}{(2x+1)^2\sqrt{2x+1}}$$

$$= -6(2x+1)^{-2,5}$$

2. optimisation



when $f'(x)$ is positive, the graph of $f(x)$ is increasing
when $f'(x)$ is negative, — " — decreasing

<u>Important conclusion</u>: The <u>sign</u> <u>diagram</u> of $f'(x)$ determines where $f(x)$ is increasing and decreasing.

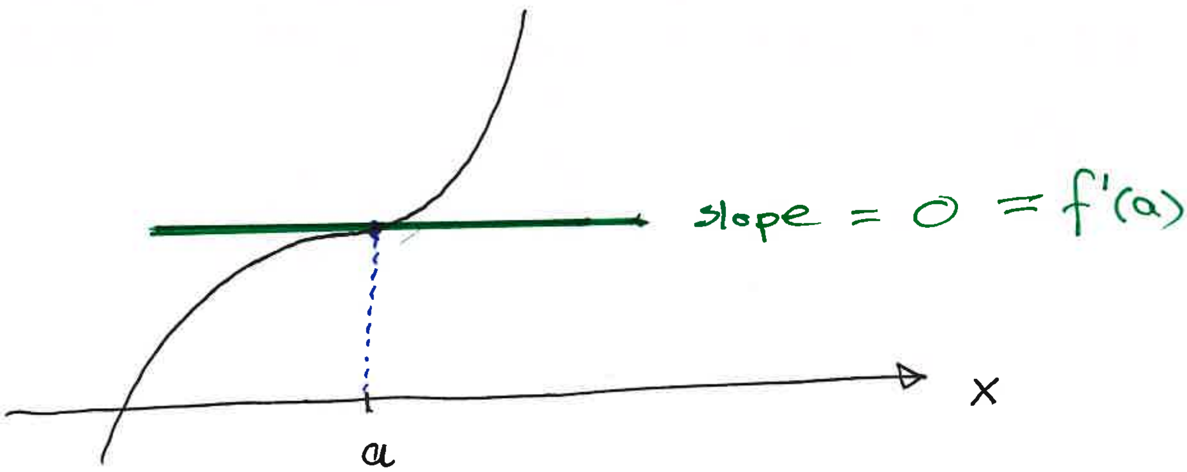
If $x=a$ is a local minimum point, then

$f'(a)=0$ <u>and</u> $f'(x)$ changes sign from $-$ to $+$

If $x=a$ is a loc. max. point, then

$f'(a)=0$ <u>and</u> $f'(x)$ changes sign from $+$ to $-$

<u>Ex</u>:



Here a is <u>neither</u> a local min. point <u>nor</u> a local max. point.

- it is a terrace point

<u>Definitions</u>: If $f'(a)=0$ then $x=a$ is a <u>stationary</u> point

Ex: $f(x) = x^3 - 6x^2 + 5$

Stationary points: Solve the equation

$f'(x) = 0$

First we have to find $f'(x)$:

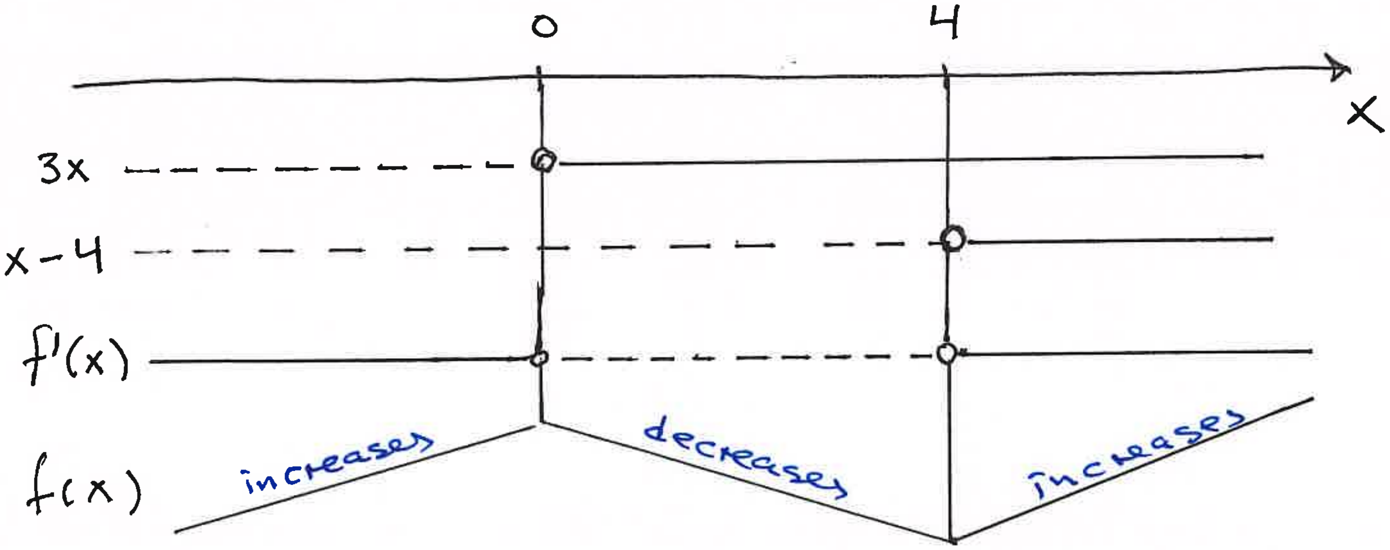
$$f'(x) = (x^3)' - 6(x^2)' + (5)'$$
$$= 3x^2 - 6 \cdot 2x + 0$$
$$= 3x^2 - 12x = 3x(x-4)$$

So $f'(x) = 0$ has the solutions

$\underline{x=0}$, $\underline{x=4}$

Where is $f(x)$ increasing / decreasing?

We determine the sign of $f'(x)$ by a sign diagram.



$f(x)$ is strictly increasing for $x \leq 0$ (in $\langle \leftarrow, 0]$)

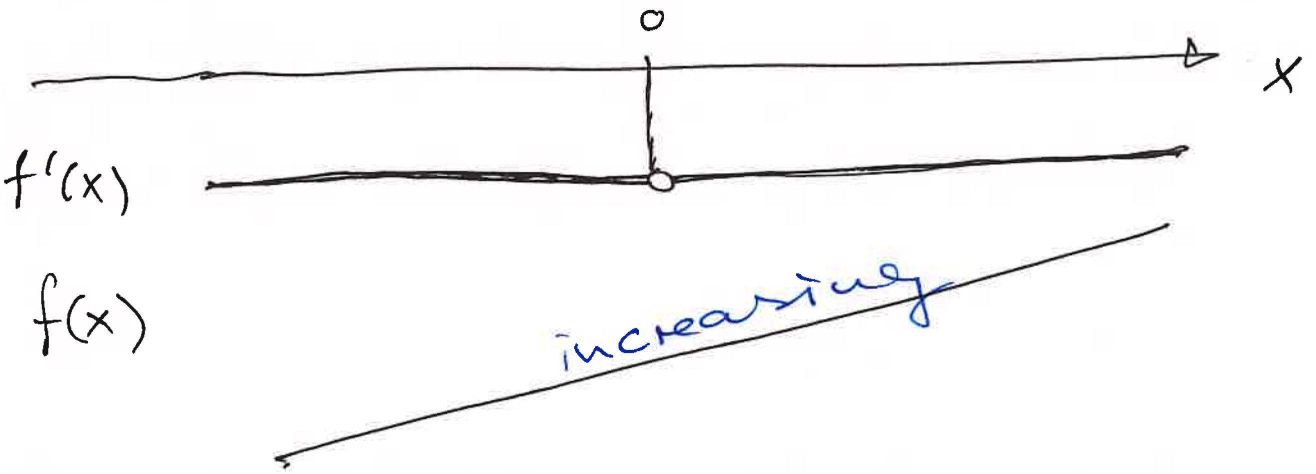
$f(x)$ is strictly decreasing for $0 \leq x \leq 4$ (in $[0, 4]$)

$f(x)$ is strictly increasing for $x \geq 4$ (in $[4, \rightarrow\rangle$)

Then $x = 0$ is a local maximum point and $x = 4$ is a local min. point.

Ex: $f(x) = x^3 + 1$

$f'(x) = 3x^2$, so $x = 0$ is a stationary point



f(x) is strictly increasing for all numbers on the number line.

The extreme value theorem: If f(x) is a continuous function on the the interval $D_f = [a, b]$ then f(x) has a global maximum and a global minimum.

Possible max/min points:

(*) stationary points ($f'(x) = 0$)

(*) cusp points (where $f'(x)$ not defined)

(*) end points (a and b).

Ex: $f(x) = x^3 - 6x^2 + 5$ and $D_f = [-1, 7]$

(*) stationary points: $f'(x) = 3x^2 - 12x = 0$

gives $\underline{x=0}$, $\underline{x=4}$

(*) cusp: non ($f'(x)$ is defined everywhere)

(*) end points: $\underline{x=-1}$, $\underline{x=7}$.

These four points are my candidate points for min/max.

$f(-1) = -2$

$f(0) = 5$

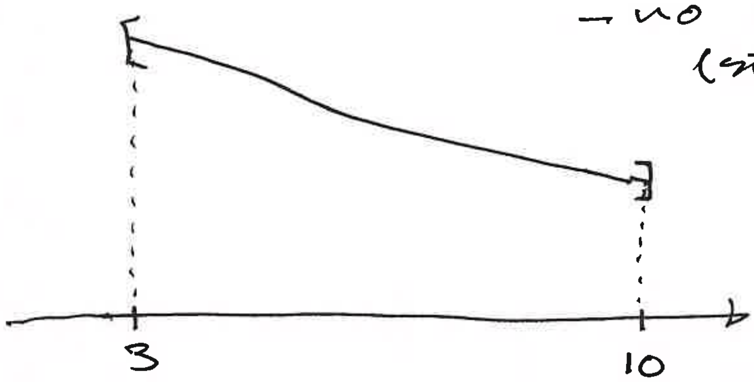
$f(4) = -27$

$f(7) = 54$

So $x=4$ gives the minimum $f(4) = -27$

and $x=7$ gives the maximum value $f(7) = 54$

Ex: $f(x) = 12 - x$ and $D_f = [3, 10]$



- no stationary points (since $f'(x) = -1$)

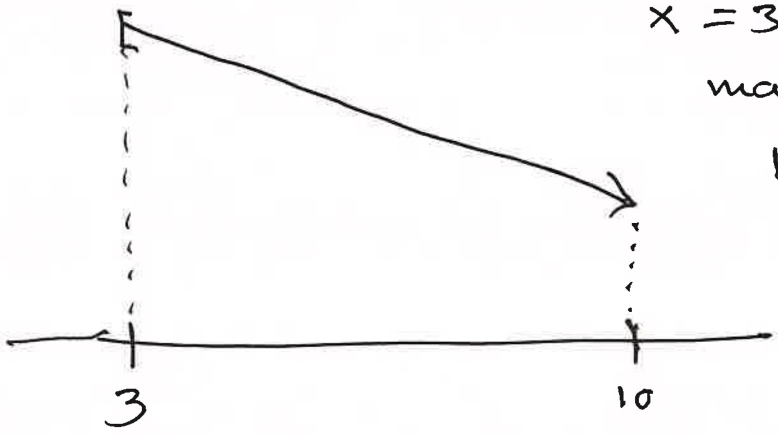
- no cusps

end points:

$x = 3$ is max point

$x = 10$ is min point.

Ex: $f(x) = 12 - x$ and $D_f = [3, 10\rangle$

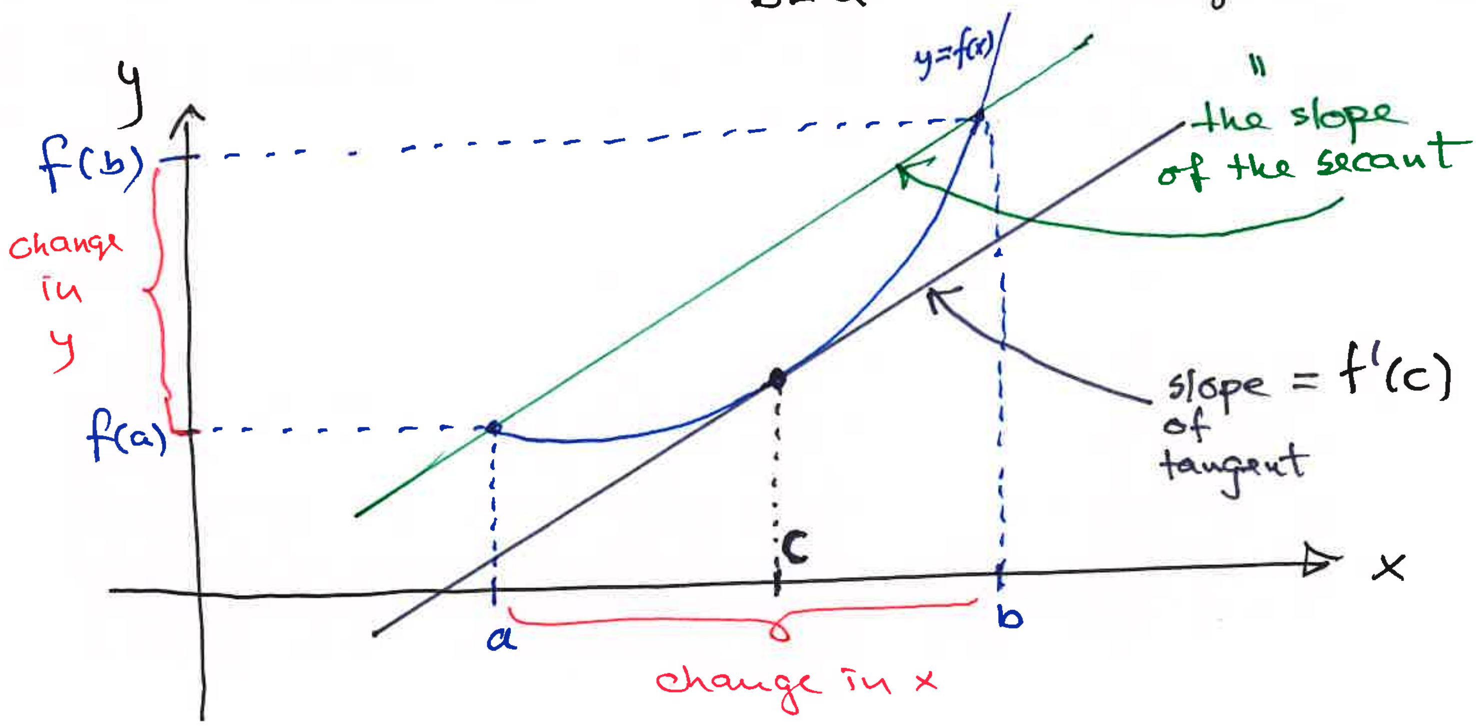


$x = 3$ still the maximum point, but no minimum point!

The mean value theorem

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable (no cusps) then there is a number c between a and b $(a < c < b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\text{change in } y}{\text{change in } x}$$



The tangent is parallel to the secant.

Ex: $f(x) = e^x + x^2$

then $f(0) = 1$ and $f(1) = e + 1$

By the mean value theorem there is a number c between 0 and 1 such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{e+1-1}{1} = e$$

Note: $f'(x) = e^x + 2x$

but we cannot find an exact solution for x to the equation $f'(x) = e$