EBA 2911, lec. 13, 4 Nov. 2019, Runar Ile

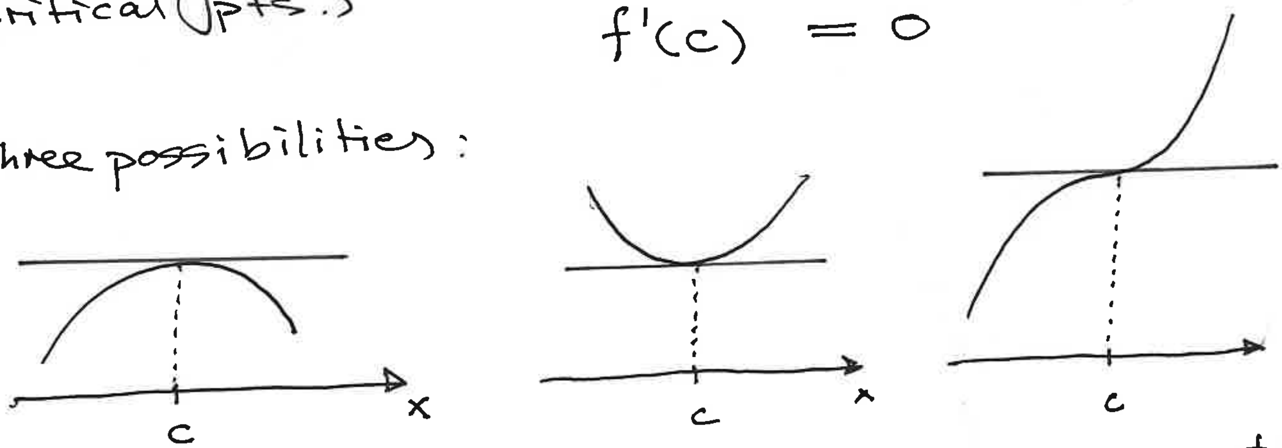
1. Repetition & problems
2. Implicit differentiation
3. The second derivative and curvature

1. Rep. & problems

Stationary points (critical pts.) : An x-value c such that

$$f'(c) = 0$$

Three possibilities:



loc. max. pt.   loc. min. pt.   terrace pt not min nor max

We use the sign diag. for $f'(x)$ to determine if $f(x)$ is

strictly increasing ($f' > 0$)

—"— decreasing ($f' < 0$)

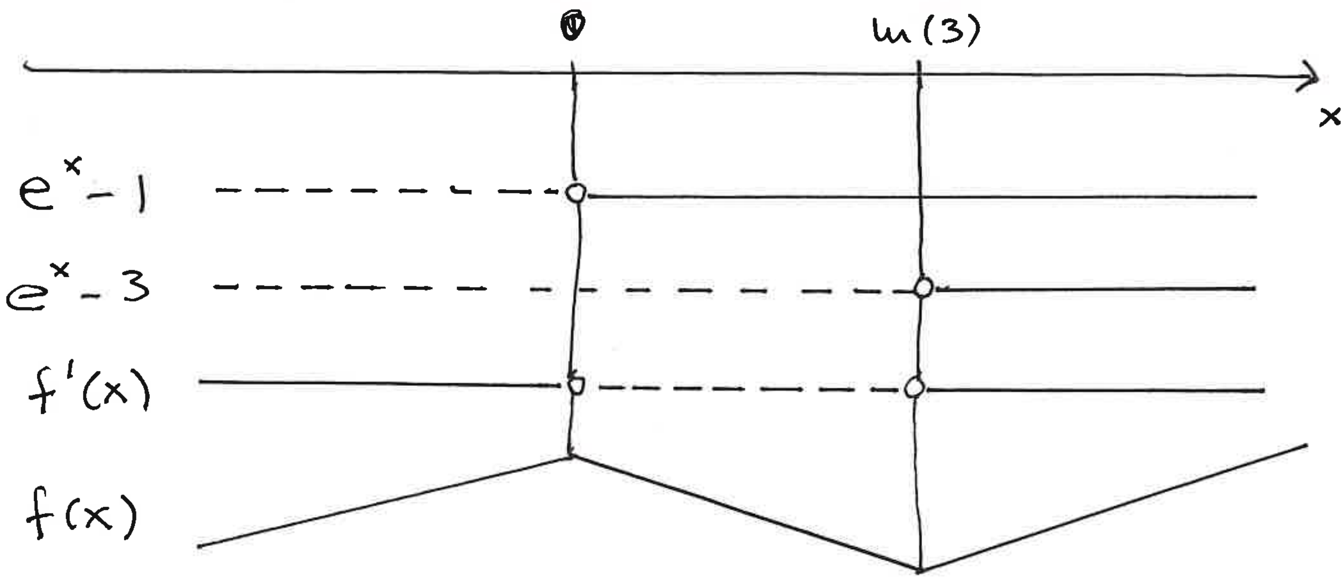
and this also determines the type of the stationary points.

Problem 4g $f'(x) = e^{2x} - 4e^{x} + 3$. When is $f(x)$ strictly increasing/decreasing? We use the sign diag. of $f'(x)$. But we have to factorise $f'(x)$ first. Put $u = e^x$

Then $u^2 = e^x \cdot e^x = e^{2x}$ so we get

$$u^2 - 4u + 3 = (u-1)(u-3) \quad \text{so}$$

$$f'(x) = (e^x - 1)(e^x - 3)$$



So $f(x)$ is strictly increasing for x in $\langle \leftarrow, 0]$

$f(x)$ —"— decreasing —"— $[0, \ln(3)]$

$f(x)$ —"— increasing —"— $[\ln(3), \rightarrow\rangle$

$\boxed{\text{The extreme value theorem}}$

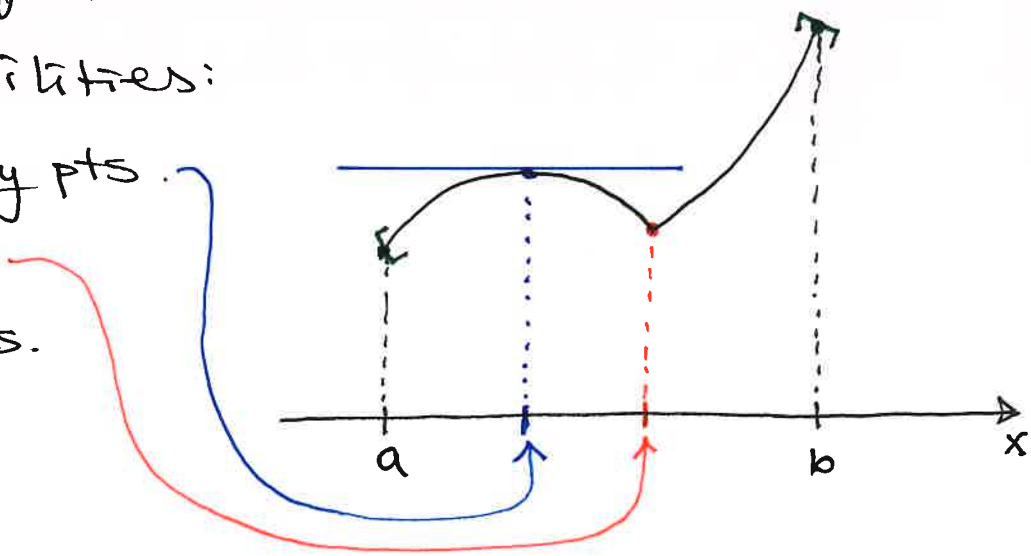
If the domain of definition is a closed interval $[a, b]$ then $f(x)$ has a (global) max. and a (glob) min on the interval.

The possibilities:

(*) Stationary pts.

(*) Cusps

(*) End pts.



To find the max. and the min. we have to compute the y-values of those x-values by $f(x)$.

<u>Prob. 5f</u> $f(x) = \ln(1+e^{-x})$, $D_f = [4, 5]$

would like to find max/min.

Stationary pts: Solve the eq. $f'(x) = 0$

To find $f'(x)$ we use the chain rule twice.

Put $u = 1 + e^{-x}$. Get $g(u) = \ln(u)$

$g'(u) = \frac{1}{u}$

small intermediate calc.
$$\left\{ (e^{-x})' \underset{u=-x}{\overset{\text{chain r. with}}{=}} (-x)'_x \cdot (e^u)'_u = -1 \cdot e^u = -e^{-x} \right.$$

Then $u'(x) = 0 - e^{-x} = -e^{-x}$

Then $f'(x) \overset{\text{chain r.}}{=} \frac{1}{u} \cdot (-e^{-x}) = -\frac{e^{-x}}{1+e^{-x}}$

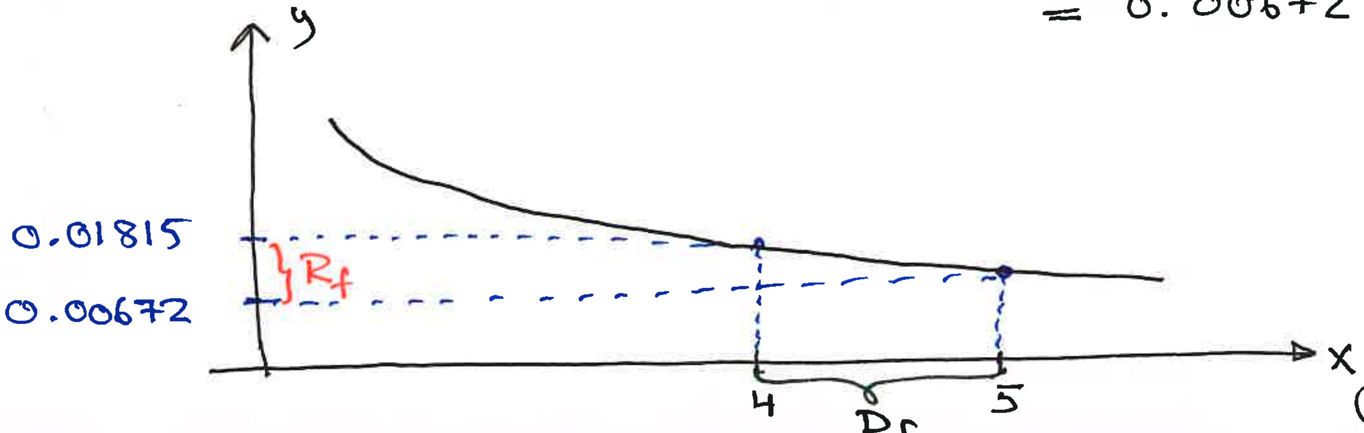
which is negative for all x.

There are no cusps ($f'(x)$ defined for all x)

Then the end pts. give the extreme values.

$x = 4$ gives the max. $f(4) = \ln(1+e^{-4})$
$= \underline{\underline{0.01815}}$

$x = 5$ gives the min. $f(5) = \ln(1+e^{-5})$
$= 0.00672$



Prob. 3C Easier to determine what is wrong!

Suppose $f(x)$ is the green one. Then the violet one is $f'(x)$. But the slop of the green one is negative for $x > 3$ while the violet one is bigger than 1. Hence $f(x)$ is the violet one.

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2. Implicit differentiation

Ex: $f(x) = \frac{1}{x} = x^{-1}$

$$f'(x) = (-1)\cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

- usual differentiation.

Instead: Put $y = f(x)$, so $y = \frac{1}{x}$ $\mid \cdot x$

we get $xy = 1$

We differentiate each side w.r.t. x and think of y as a function of x

$$(xy)'_x = (1)'_x$$

Product rule gives

$$(x)'_x \cdot y + x\cdot(y)'_x = 0$$

$$1\cdot y + x\cdot y' = 0$$

We can solve this equation for y'.

$$x \cdot y' = -y \qquad | : x$$

$$y' = -\frac{y}{x} \qquad \left(= -\frac{\frac{1}{x}}{x} = -\frac{1}{x^2} \right)$$

The point is that we don't have to know the expression for $y(x)$

This is called implicit differentiation.

We can actually use this to find slopes:

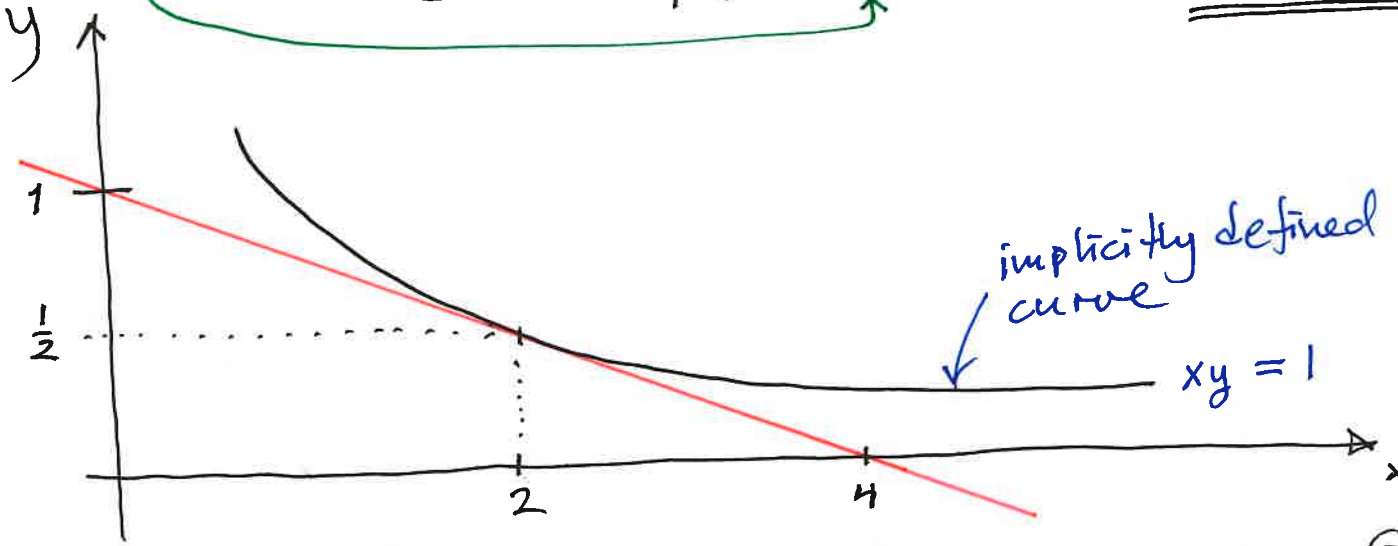
E.g. if $\underline{x = 2}$, then $2 \cdot y = 1$ so

$$\underline{y = \tfrac{1}{2}}$$

also: $$y'\Big|_{\substack{x=2 \\ y=\frac{1}{2}}} = -\frac{\frac{1}{2}}{2} = \underline{\underline{-\frac{1}{4}}}$$

We can use this slope to find the function expression $h(x)$ for the tangent in the point $(2, \frac{1}{2})$ by the point-slope formula:

$$h(x) - \frac{1}{2} = -\frac{1}{4}(x - 2), \text{ so } \underline{\underline{h(x) = -\tfrac{1}{4}x + 1}}$$



Problem A curve is implictly defined by the equation $y^2 - x^3 = 1$

a) Express y' by x and y using implicit differentiation (think of y as a function of x)

b) Find all solutions for y when $x = 2$

c) compute y' for these points.

Solution:

a) $$(y^2)'_x - (x^3)'_x = (1)'_x$$

$$2y \cdot y' - 3x^2 = 0$$

and solve for y': $2yy' = 3x^2$

$$y' = \frac{3x^2}{2y}$$

Chain rule:
$u = y$
$g(u) = u^2$
$u'_x = y'_x$
$g'(u) = 2u$
$= 2y$

b) $x = 2$. Solve $y^2 - 2^3 = 1$ for y.

$$y^2 = 9$$
$$y = \pm 3$$

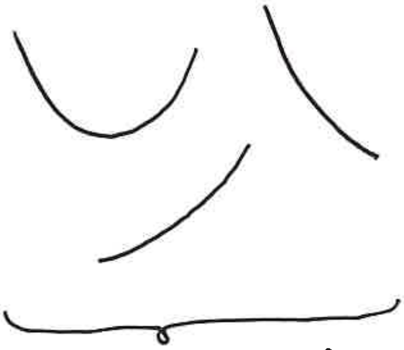
c) $(2, -3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot (-3)} = -2$

$(2, 3)$: $y' = \frac{3 \cdot 2^2}{2 \cdot 3} = 2$

3. The second order derivative and curvature

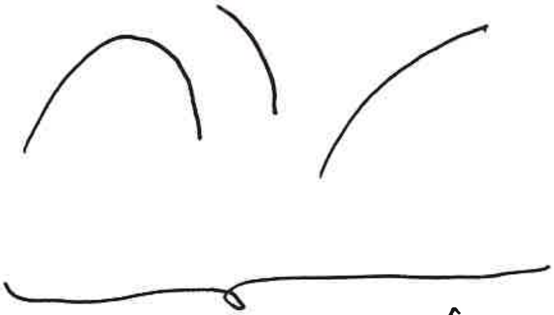
In which direction does the graph bend?

<u>bending up</u>

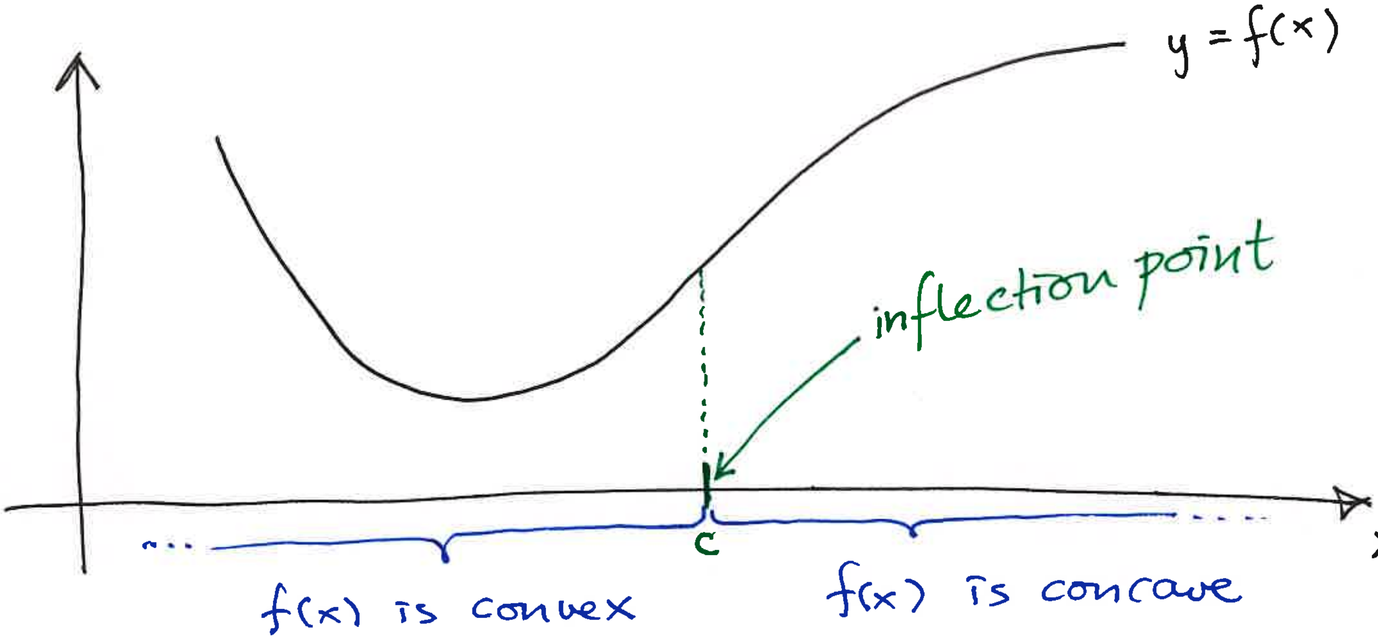


the graphs of <u>convex</u> functions

<u>bending down</u>



the graphs of <u>concave</u> functions



<u>Definition</u>: $f(x)$ is convex (or concave) in the interval $[a, b]$ if

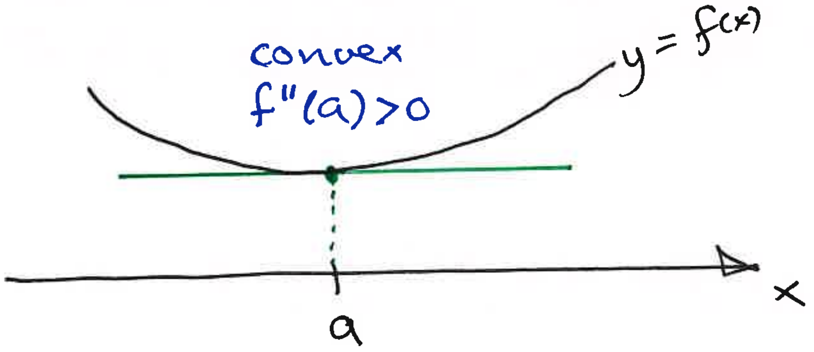
$$f''(x) \geq 0 \text{ for all } x \text{ in } \langle a, b \rangle$$

$$(\text{or } f''(x) \leq 0 \text{ for all } x \text{ in } \langle a, b \rangle)$$

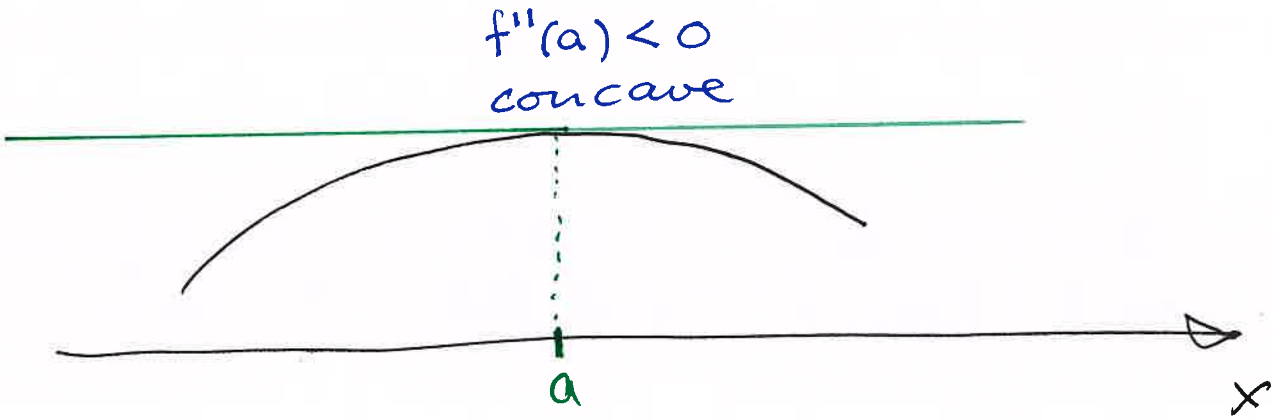
The number c is an inflection point if $f''(x)$ changes sign at $x = c$.

Second derivative test (p 308)

Suppose $x=a$ is a stationary point for $f(x)$. If $f''(a)>0$ then $x=a$ is a local minimum point



If $f''(a) < 0$ then $x=a$ is a local maximum point.



Ex: $f(x) = x^3 - 3x^2 + 5$

Then $f'(x) = 3x^2 - 6x$

Stationary points : solutions to $3x^2 - 6x = 0$

that is $3x(x-2) = 0$ so $\underline{x=0}$ or $\underline{x=2}$

We use the sec. der. test:

$f''(x) = [f'(x)]' = [3x^2 - 6x]' = 6x - 6$

Insert: $f''(0) = 6\cdot 0 - 6 = -6 < 0$ so $x=0$ loc. max. pt.

$f''(2) = 6\cdot 2 - 6 = 6 > 0$ so $x=2$ is a loc. min pt.

Convex optimisation

If $f(x)$ is convex everywhere in its domain, then any stationary point will be a global minimum point.

(and if $f(x)$ is concave everywhere : global max point)

Ex: $f(x) = 3x^2 - 2x + 1$

$f'(x) = 6x - 2$

so $x = \frac{1}{3}$ is the only stationary point

$f''(x) = 6 > 0$ then $f(x)$ is convex everywhere and

$x = \frac{1}{3}$ is a global minimum point.