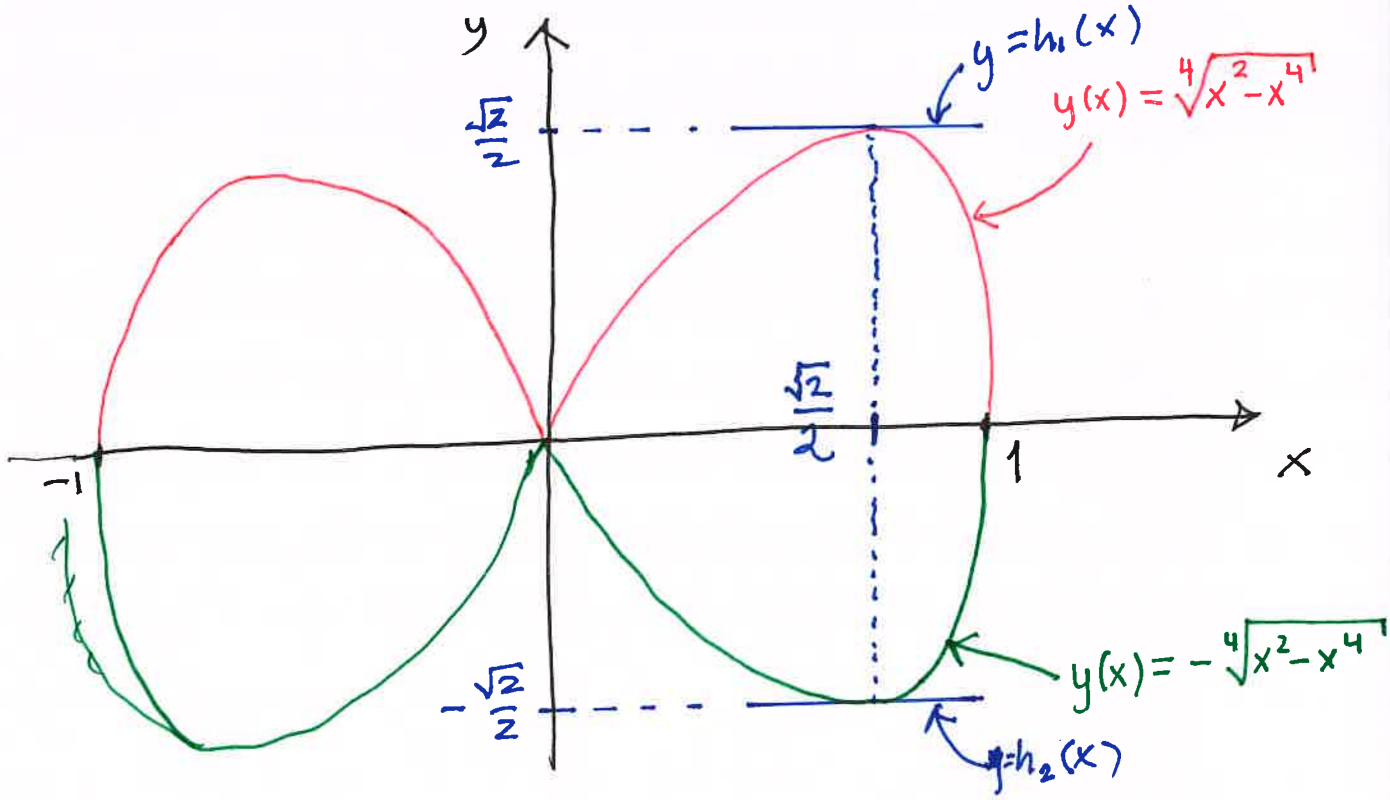
EBA2911, lecture 14, 11 Nov. 2019, Runar Ile

1. Repetition and problems
2. l'Hôpital's rule
3. Marginal cost, average unit cost, marginal revenue
4. Elasticity

1. Rep. & problems

Implicit differentiation: We have a curve which is defined by an equation in two variables. We want to find the slope of a tangent to this curve without finding the expression of the function.

Problem 1c $\quad x^4 - x^2 + y^4 = 0 \quad (*)$



We think of y as a function of x

Find $y'(x)$ expressed by $y(x)$ and x

Differentiate each side of (*) w.r.t x :

$$(x^4)'_x - (x^2)'_x + (y^4)'_x = (0)'_x$$

power rule + chain rule

$$4x^3 - 2x + 4y^3 \cdot y' = 0$$

solve this equation for y'

$$4y^3 \cdot y' = 2x - 4x^3 = 2x(1-2x^2)$$

$$y'(x) = \frac{x(1-2x^2)}{2y^3}$$

Find the possible y-values for $x = \frac{\sqrt{2}}{2}$ from eq. (*). Then

$$x^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} \quad \text{and} \quad x^4 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So (*) becomes $\frac{1}{4} - \frac{1}{2} + y^4 = 0$

that is $y^4 = \frac{1}{4}$

that is $y^2 = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$

and then $y = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}}$

$$= \pm\frac{\sqrt{2}}{2}.$$

The slope of the tangents :

$$y'\Big|_{\substack{x=\sqrt{2}/2 \\ y=\pm\sqrt{2}/2}} = \frac{\frac{\sqrt{2}}{2}\cdot\left(1-2\left(\frac{\sqrt{2}}{2}\right)^2\right)}{2\cdot\left(\pm\frac{\sqrt{2}}{2}\right)^3} = \underline{\underline{0}}$$

Then the tangent functions are constant

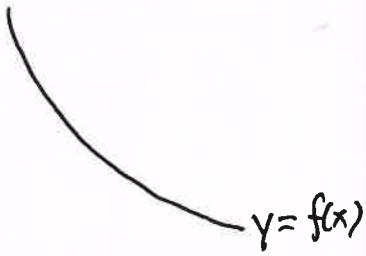
$$\underline{\underline{h_1(x) = \frac{\sqrt{2}}{2}}} \quad \text{and} \quad \underline{\underline{h_1(x) = -\frac{\sqrt{2}}{2}}}$$

<u>Curvature</u>

Convex: The graph bends upwards

so $f'(x)$ is increasing

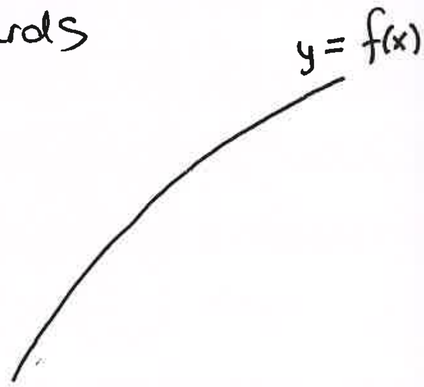
so $f''(x) \geqslant 0$



Concave: The graph bends downwards

so $f'(x)$ is decreasing

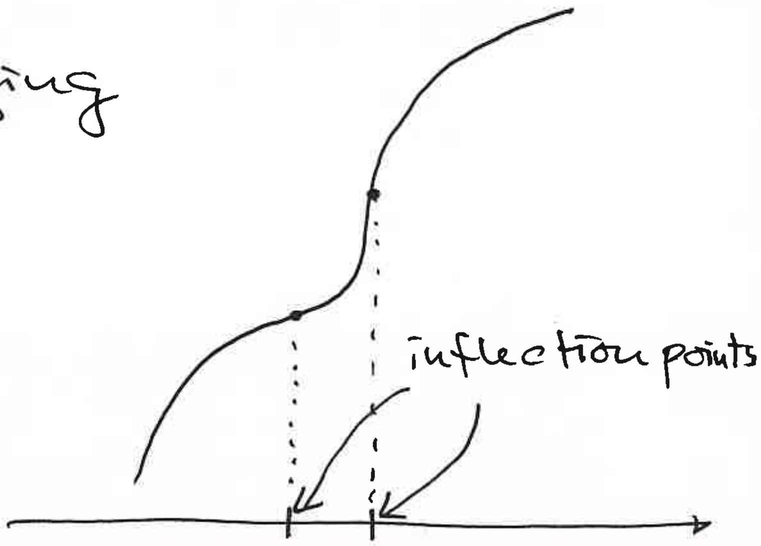
so $f''(x) \leqslant 0$



Inflection point :

where the sign of

$f''(x)$ is changing



Point-slope formula: $y - y_0 = a(x - x_0)$

for a line through the point (x_0, y_0) with slope a.

Problem 6c: See It's Learning.

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2. l'Hôpital's rule

Limits of the type $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$.

Notation: $\lim\limits_{x \to 5} f(x)$ is the number which $f(x)$ is approaching when x is approaching 5 more and more.

Ex: $f(x) = \dfrac{3x - 3}{\ln(x)}$. Want to find $\lim\limits_{x \to 1} f(x)$.

Numerator: $3x - 3 \xrightarrow[x \to 1]{} 3 \cdot 1 - 3 = 0$

Denominator: $\ln(x) \xrightarrow[x \to 1]{} \ln(1) = 0$

$\frac{0}{0}$ - expression

Then we can use l'Hôpital's rule to proceed:

$$\lim_{x \to 1} f(x) \overset{\text{l'Hôp.}}{=} \lim_{x \to 1} \frac{(3x-3)'}{[\ln(x)]'} = \lim_{x \to 1} \frac{3}{\frac{1}{x}}$$

$$= \frac{3}{\frac{1}{1}} = \underline{\underline{3}}$$

Note: Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Problem: Use l'Hôpital's rule to determine the limit

$$\lim_{x \to 0} \frac{x}{e^x - 1}$$

Solution: Numerator: $x \xrightarrow[x\to 0]{} 0$

Denominator: $e^x - 1 \xrightarrow[x\to 0]{} e^0 - 1 = 1 - 1 = 0$

So we have a $\frac{0}{0}$-situation.

Then we can apply l'Hôpital's rule:

$(x)' = 1$ and $(e^x - 1)' = e^x$

$$\lim_{x \to 0} \frac{x}{e^x - 1} \overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{1}{e^x} = \frac{1}{e^0} = \frac{1}{1} = \underline{\underline{1}}$$

Ex: $$\lim_{x \to \infty} \underbrace{\frac{x^2}{e^x}}_{\frac{\infty}{\infty}} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \underbrace{\frac{2x}{e^x}}_{\frac{\infty}{\infty}} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \frac{2}{e^x} = \underline{\underline{0}}$$

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3. Marginal cost, average unit cost, marginal revenue

$C(x)$ is cost of producing x units of some commodity

$C'(x)$ is the marginal cost

Interpretation: The cost of producing one more unit than x units.

$$= C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1} \approx \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} = C'(x)$$

Why $C'(x)$? – much simpler to work with!

$R(x)$ revenue by selling x units

$R'(x)$ marginal revenue

Ex: x = tons of salmon

$R'(50)$ = extra revenue by selling 1 extra ton of salmon more than 50 ton.

The profit function:

$$P(x) = R(x) - C(x)$$

Economists: $\pi(x)$

$P'(x)$ = the marginal profit function

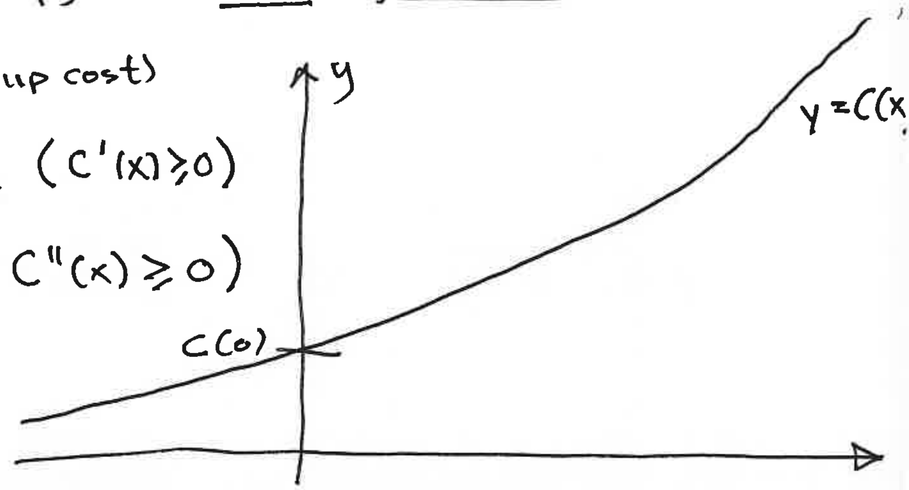
Average unit cost by producing x units:

$$A(x) = \frac{C(x)}{x}$$

"cost per unit" – not a constant!

Definition $C(x)$ is a cost function if

1. $C(0) > 0$ (start-up cost)
2. $C(x)$ increasing ($C'(x) \geq 0$)
3. $C(x)$ convex ($C''(x) \geq 0$)



Definition: If $x = c$ is a minimum point for $A(x)$ then c is called the cost optimum (the minimal average unit cost)

Result: If $C''(x) > 0$, then the cost optimum is the solution of the equation $C'(x) = A(x)$

Reason: we determine the stationary points of $A(x)$:

$$A'(x) = \left(\frac{C(x)}{x}\right)'$$

quot.r.

$$= \frac{C'(x) \cdot x - C(x) \cdot 1}{x^2} \quad \left| \frac{:x}{:x} \right.$$

$$= \frac{C'(x) - A(x)}{x}$$

so $A'(x) = 0$ is equivalent to $C'(x) = A(x)$

Assume $x = c$ is such a stationary point.

Use second derivative test:

If $A''(c) > 0$ then c is a (loc.) min. point

$$A''(x) = \frac{[C'(x) - A(x)]' \cdot x - [C'(x) - A(x)] \cdot 1}{x^2}$$

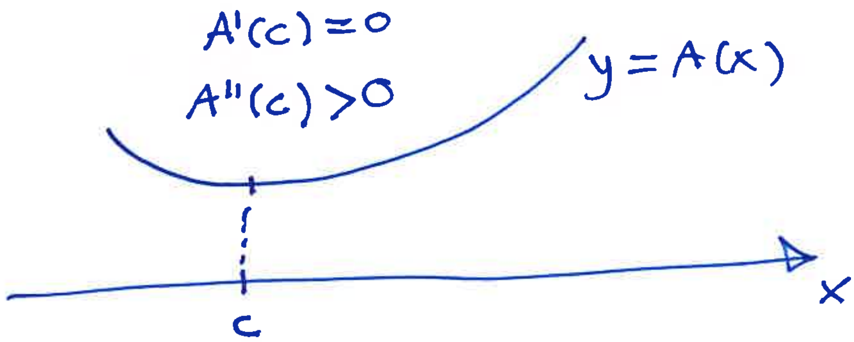
$$= \frac{[C''(x) - A'(x)] \cdot x - [C'(x) - A(x)]}{x^2}$$

Substitute $x = c$

$$A''(c) = \frac{[C''(c) - \overbrace{A'(c)}^{=0}] \cdot c - \overbrace{[C'(c) - A(c)]}^{=0}}{c^2}$$

$$= \frac{C''(c) \cdot \cancel{c}}{c^{\cancel{2}}} = \frac{C''(c)}{c} > 0$$

(for $c > 0$)



Ex: $C(x) = x^2 + 200x + 160\,000$

This is a cost function because:

① $C(0) = 160000 > 0$

② $C'(x) = 2x + 200 > 0$ for $x \geq 0$.

③ $C''(x) = 2 > 0$

Average unit cost is the solution of

$$C'(x) = A(x) = \frac{C(x)}{x} = \frac{x^2 + 200x + 160000}{x}$$

$$= x + 200 + \frac{160000}{x}$$

so the equation is

$$2\cancel{x}\cancel{+200} = \cancel{x}\cancel{+200} + \frac{160\,000}{x}$$

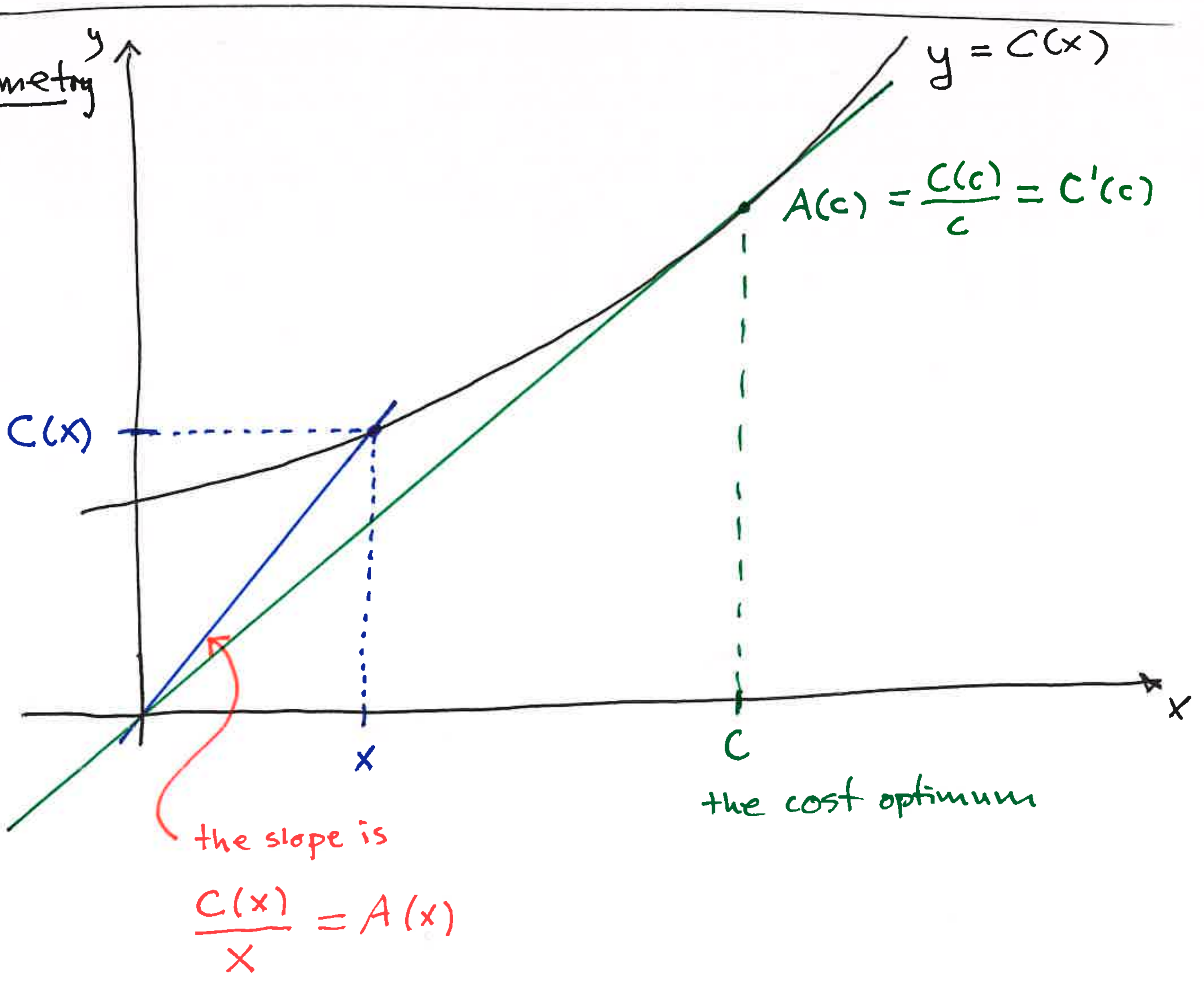
$$x = \frac{160\,000}{x} \quad \text{that is} \quad x^2 = 160\,000$$

so $\underline{\underline{x = 400}}$ (only pos. x)

The minimal average unit cost

is $A(400) = C'(400) = 2 \cdot 400 + 200 = \underline{\underline{1000}}$

<u>Geometry</u>



and $A(c) = \dfrac{C(c)}{c}$ is the minimal average unit cost.

4. Elasticity

p = price/unit

$D(p)$ = demand with price p

(= # sold units)

The problem of comparing different units.

Ex: A barrel North Sea crude oil costs \$66.42

1 litre of — " — NOK 3.55

The price elasticity of the demand is

$$\varepsilon = \frac{\text{relative change in demand}}{\text{relative change in price}}$$

Ex: In a month the price of a commodity drops from 12 thousand to 10 thousand, and ~~the~~ the demand increases from 50 mill to 60 mill.

Then

$$\varepsilon = \frac{\left(\frac{60-50}{50}\right)}{\left(\frac{10-12}{12}\right)} = \frac{\frac{10}{50}}{-\frac{2}{12}} = \frac{120}{-100} = \underline{\underline{-1,2}}$$

(if the price increases by 1%, the demand decreases by 1,2%).

Suppose the price is changed from p to $p+h$. Then the relative price change is

$$\frac{p+h-p}{p} = \frac{h}{p}$$

$$= \frac{\dfrac{\text{rel. change of demand}}{\text{rel. change of price}}}{} $$

$$= \frac{\dfrac{D(p+h) - D(p)}{D(p)}}{\dfrac{h}{p}}$$

$$= \frac{D(p+h) - D(p)}{h} \cdot \frac{p}{D(p)}$$

$$\Big\downarrow \; h \to 0$$

$$\varepsilon(p) = D'(p) \cdot \frac{p}{D(p)}$$

– the momentary price elasticity of the demand function.

Revenue $R(p) = p \cdot D(p)$

The marginal revenue w.r.t. price is

$$R'(p) \overset{\text{product r.}}{=} 1 \cdot D(p) + p \cdot D'(p)$$

$$= D(p)\left[1 + \frac{p \cdot D'(p)}{D(p)}\right]$$

$$= \underbrace{D(p)}_{\text{always pos.}}\underbrace{[1 + \varepsilon(p)]}_{\text{pos/neg. ??}}$$

If $\varepsilon(p) < -1$
we get neg. $R'(p)$
- get elastic demand

If $\varepsilon(p) > -1$
we get pos. $R'(p)$
- we get inelastic demand

If $\varepsilon(p) = -1$
the demand is unit elastic

Ex: $D(p) = 50 - p$ for $0 < p < 50$

Then $D'(p) = -1$ and $\varepsilon(p) = \dfrac{D'(p) \cdot p}{D(p)}$

$$= \frac{(-1) \cdot p}{50 - p} = \frac{-p}{50-p}$$

When do we have elastic demand?

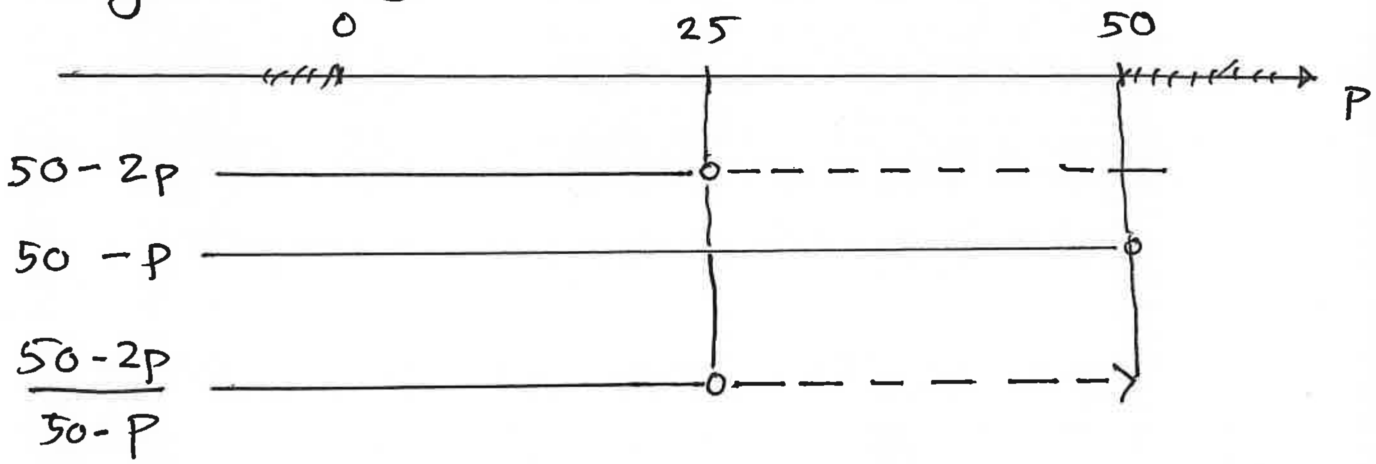
Solves the inequality $\dfrac{-p}{50-p} < -1$

$$\frac{-p}{50-p} + 1 < 0$$

that is $$\frac{-p + 50 - p}{50 - p} < 0$$

that is $$\frac{50 - 2p}{50 - p} < 0$$

Sign diagram



So elastic demand for p in $\langle 25, 50 \rangle$

inelastic —— " —— $\langle 0, 25 \rangle$

unit elastic —— $p = \underline{\underline{25}}$.