EBA2911, lecture 15, 15 Nov. 2019, Runar Ile

1. Repetition and problems
2. Linear approximation
3. Quadratic approximation
4. Taylor polynomials

1. Rep. & problems

l'Hôpital's rule: Used for limits $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

- Differentiate the numerator and the denominator seperately.
- Consider the limit of the new quotient.

Problem 1h

$$\lim_{x \to 1} \frac{\ln(x)}{\sqrt{x}-1} \overset{\text{l'Hôp}}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{1}{\frac{1}{2}} = \underline{\underline{2}}$$

$[\ln(x)]' = \frac{1}{x}$

$(\sqrt{x}-1)' = (x^{\frac{1}{2}}-1)'$

$= \frac{1}{2}x^{-\frac{1}{2}} - 0 = \frac{1}{2\sqrt{x}}$

$\lim_{x \to \infty}$ –"– "$\frac{\infty}{\infty}$"

$$= \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \xrightarrow[x \to \infty]{} 0$$

so the line $y=0$ (the x-axis) is a horizontal asymptote for $\frac{\ln(x)}{\sqrt{x}-1}$

Cost functions: Three conditions

1. $C(0) > 0$
2. $C(x)$ is increasing ($C'(x) \geq 0$)
3. $C(x)$ is convex ($C''(x) \geq 0$)

Average unit cost $\quad A(x) = \dfrac{C(x)}{x}$

Cost optimum : Minimum point for $A(x)$.

- it is (also) the solution of the equation $\quad A(x) = C'(x)$

if $\quad C''(x) > 0$

Problem 3d $\quad C(x) = 1000 \cdot e^{\overbrace{0.0004 \cdot (x+5)^2}^{u = u(x)}}$

Is a cost function because:

① $\quad u(0) = 0.0004 \cdot 5^2 = 0.01$ so

$$c(0) = 1000\, e^{0.01} = 1010.05 > 0$$

② $\quad u'(x) = 0.0004 \cdot 2(x+5) \cdot 1 = 0.0008\,(x+5)$

$$g(u) = 1000\, e^u \quad , \quad g'(u) = 1000 \cdot e^u$$

so $\quad C'(x) = g'(u) \cdot u'(x)$

$$= 1000\, e^u \cdot 0.0008 \cdot (x+5) > 0 \quad \text{for } x > -5$$

$$= 0.8\,(x+5) \cdot e^u$$

Hence $C(x)$ is increasing for $x \geq 0$.

③ $\quad c''(x) \overset{\text{product r.}}{=} [0.8(x+5)]' \cdot e^u + 0.8(x+5) \cdot (e^u)'$

$$\overset{\text{chain r.}}{=} 0.8\, e^u + 0.8(x+5) \cdot e^u \cdot 0.0008\,(x+5)$$

$$= 0.8\,(1 + 0.0008\,(x+5)^2)\, e^u > 0$$

for all x.

So $C(x)$ is convex.

The cost optimum is the solution of eq. $C'(x) = A(x) = \frac{C(x)}{x}$

(because $C''(x) > 0$)

So:

$$0.8(x+5)e^{u} = \frac{1000 \cdot e^{u}}{x} \qquad | \cdot x$$

$$0.8\,x(x+5)\,e^{u} = 1000\,e^{u} \qquad | : e^{u}$$

$$0.8\,x(x+5) = 1000 \qquad | : 0.8$$

$$x^2+5x = x(x+5) = \frac{1000}{0{,}8} = 1250$$

$$\left(x+\frac{5}{2}\right)^2 = 1250 + 6.25 = 1256.25$$

$$x + \frac{5}{2} = \sqrt{1256.25} \qquad (\text{since } x \geq 0)$$

$$x = -\frac{5}{2} + \sqrt{1256.25} = \underline{\underline{32.94}}$$

Minimal unit cost:

$$A(32.94) = C'(32.94) = 0.8(32.94+5)e^{0.0004\cdot\left(\frac{32.94}{\;+\;5}\right)^2}$$

$$= \underline{\underline{53.99}}$$

The price elasticity of the demand:

$$\varepsilon = \frac{\text{relative change in demand}}{\text{relative change in price}}$$

If p = price and $D(p)$ = demand as a function of p, then

$$\mathcal{E}(p) = \frac{D'(p)\cdot p}{D(p)}$$

<u>Problem 7a</u> $D(p) = 100 - 2p \quad (0 < p < 50)$

$D'(p) = -2$ and $\mathcal{E}(p) = \dfrac{-2p}{100-2p}$

Elastic demand: $\mathcal{E}(p) < -1$ – for which p?

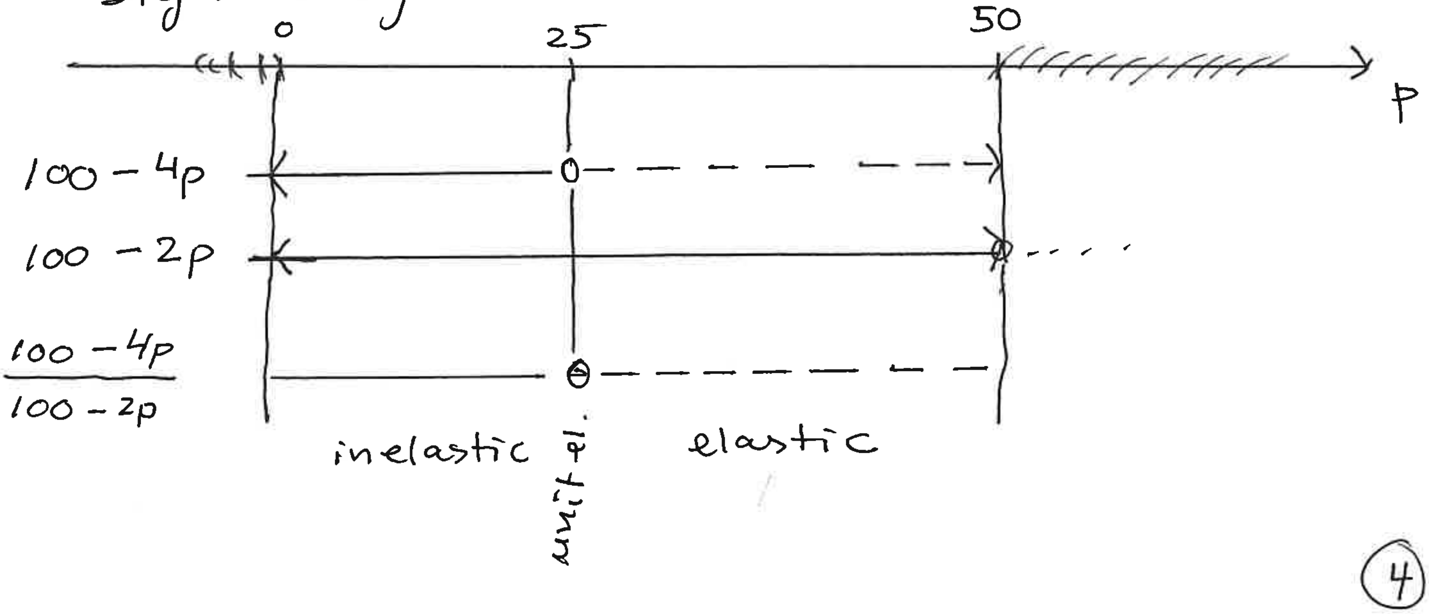
that is: $\dfrac{-2p}{100-2p} < -1$

that is: $\dfrac{-2p}{100-2p} + 1 < 0$

so $\dfrac{-2p + 100 - 2p}{100 - 2p} < 0$

so $\dfrac{100 - 4p}{100 - 2p} < 0$

sign diagram:



Elastic demand : $25 < p < 50$

Inelastic —"— : $0 < p < 25$

Unit elastic demand : $p = 25$.

Interpretation : If the price increases (from p) by 1 %, the demand changes by $\varepsilon(p)$ %

Ex : $\varepsilon(30) = \frac{-2 \cdot 30}{100 - 2 \cdot 30} = \frac{-60}{40} = -1{,}5$

If the price is increasing from 30 to 30.3 then the demand goes down by 1.5% from $D(30) = 40$

more interpretation :

$$R(p) = p \cdot D(p)$$

$$R'(p) = 1 \cdot D(p) + p \cdot D'(p)$$

$$= \underbrace{D(p)}_{\text{pos.}} \cdot \underbrace{[1 + \overset{-1{,}5}{\varepsilon(p)}]}_{\text{pos/neg ?}}$$

$-0{,}5$ neg!

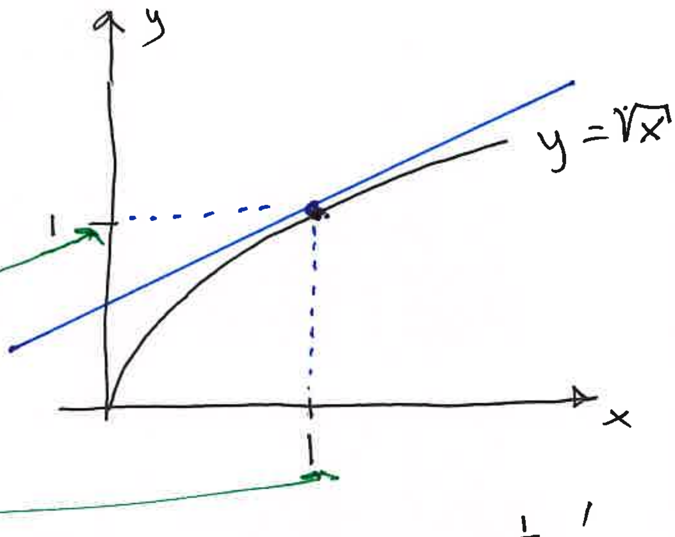
so $R(p)$ is decreasing around $p = 30$.

$R(p)$ decreasing if $\varepsilon(p) < -1$

$R(p)$ increasing if $\varepsilon(p) > -1$

2. Linear approximation

Ex: $f(x) = \sqrt{x}$ $(x \geq 0)$



By the point-slope formula the tangent line at 1 is

$$y - 1 = f'(1)(x - 1)$$

$$y = \underbrace{1}_{=f(1)} + f'(1)(x-1)$$

$$= 1 + \frac{1}{2}(x-1).$$

$$f'(x) = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

This is a function in x denoted

$$P_1(x) = 1 + \frac{1}{2}(x-1).$$

It is called the degree 1 Taylor polynomial to $\sqrt{x}$ at 1.

Ex: $P_1(1.1) = 1 + \frac{1}{2}(1.1 - 1) = 1.05$ - an approximation

$$\sqrt{1.1} = 1.04881\ldots$$

3. Quadratic approximation

Ex: $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f''(x) = \frac{1}{2}\left(x^{-\frac{1}{2}}\right)' = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$$

so $f''(1) = -\frac{1}{4\cdot 1\cdot\sqrt{1}} = -\frac{1}{4}$

$$P_2(x) = \overbrace{f(1) + f'(1)(x-1)}^{P_1(x)} + \frac{f''(1)}{2}(x-1)^2$$

$$= \underbrace{1 + \frac{1}{2}(x-1)}_{P_1(x)} - \left(\frac{\left(\frac{1}{4}\right)}{2}\right)(x-1)^2$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

Problem: Compute

$P_2(1)$, $P_2'(1)$ and $P_2''(1)$.

Solution: $P_2(1) = 1 + \frac{1}{2}(1-1) - \frac{1}{8}(1-1)^2$

$$= 1 + \frac{1}{2} \cdot 0 - \frac{1}{8} \cdot 0 = 1$$

$$= \sqrt{1} = f(1).$$

$$P_2'(x) = (1)' + \frac{1}{2}(x-1)' - \frac{1}{8}\left[(x-1)^2\right]'$$

$$= 0 + \frac{1}{2} \cdot 1 - \frac{1}{8} \cdot 2 \cdot (x-1)$$

$$= \frac{1}{2} - \frac{1}{4}(x-1)$$

$$P_2'(1) = \frac{1}{2} - \frac{1}{4}(1-1) = \frac{1}{2} - \frac{1}{4} \cdot 0$$

$$= \frac{1}{2} = f'(1)$$

$$P_2''(x) = \left(\frac{1}{2}\right)' - \frac{1}{4}\left[(x-1)\right]' = 0 - \frac{1}{4} \cdot 1$$

$$= -\frac{1}{4}$$ so

$$P_2''(1) = -\frac{1}{4} = f''(1)$$

$P_2(x)$ is the second degree Taylor polynomial of $f(x)=\sqrt{x}$ in 1.

Pattern: $\boxed{P_2(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2}$

(so $a = 1$ in the example)

$$\sqrt{2} = f(2) \approx P_2(2) = 1 + \tfrac{1}{2}(2-1) - \tfrac{1}{8}(2-1)^2$$
$$= 1 + \tfrac{1}{2}\cdot 1 - \tfrac{1}{8}\cdot 1^2 = \frac{11}{8} = 1.375$$

$(\sqrt{2} = 1.41421)$

4. Taylor polynomials

<u>Ex</u>: $f(x) = \sqrt{x}$ in 1 :

$$P_1(x) = 1 + \tfrac{1}{2}(x-1)$$

$$P_2(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2$$

$$P_3(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \frac{f'''(1)}{6}\cdot(x-1)^3$$

$$f'''(x) = \Big(\underbrace{-\tfrac{1}{4}x^{-\frac{3}{2}}}_{f''(x)}\Big)' = -\tfrac{1}{4}\cdot\left(-\tfrac{3}{2}\right)\cdot x^{-\frac{3}{2}-1}$$

$$= \frac{3}{8}\cdot x^{-\frac{5}{2}} = \frac{3}{8x^2\sqrt{x}}$$

$$f'''(1) = \frac{3}{8\cdot 1^2\sqrt{1}} = \frac{3}{8}$$

$$P_3(x) = 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \frac{3}{8.6}(x-1)^3$$

And $$P_3(2) = 1 + \tfrac{1}{2}(2-1) - \tfrac{1}{8}(2-1)^2 + \frac{3}{48}(2-1)^3$$

$$= 1 + \frac{1}{2} - \frac{1}{8} + \frac{3}{48} = \frac{69}{48}$$

$$= 1.4375$$

$$(\sqrt{2} = 1.41421\ldots)$$

Pattern: $$\boxed{P_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3}$$

The degree n Taylor polynomial for $f(x)$ in a is

$$\boxed{\begin{array}{l} P_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2 + \ldots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n \\[2ex] \text{where } n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1. \end{array}}$$

<u>Problem</u>: Determine the second degree Taylor polynomial of $f(x) = \sqrt{x}$ in 4 (the $P_2(x)$)

Solution: $f'(x) = \dfrac{1}{2\sqrt{x}}$ so $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$f''(x) = -\dfrac{1}{4x\sqrt{x}}$ so $f''(4) = -\dfrac{1}{4\cdot 4\sqrt{4}} = -\dfrac{1}{32}$

Then $$P_2(x) = 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2$$