Plan

1. Introduction to integration and definite integrals
2. Antiderivatives and indefinite integrals
3. Integration rules and simple substitutions

Intro: A Integration

Topics: B Matrices and vectors

C Functions in two variables

Exercise sessions:

Thu. 14-16 in D1-065 (with norwegian students)

Mon. 14-16 in B2-050

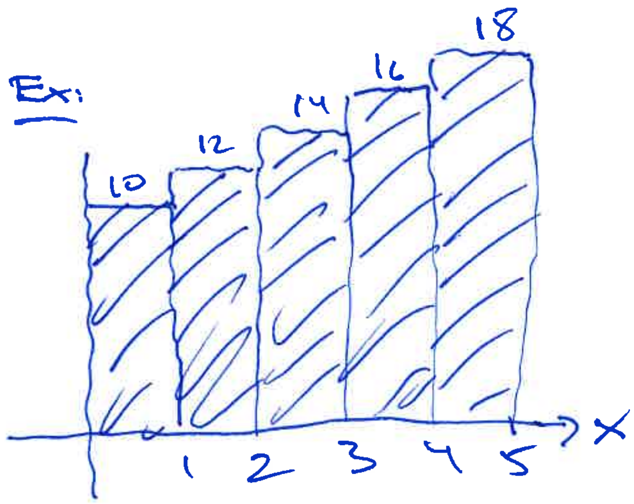
Office hours: Thu 09-11 (B4-032)

Open door policy; you can come to my office at any time

(Thu 09-11 + Mon-Wed are usually best)

① Intro to integration

Ex:

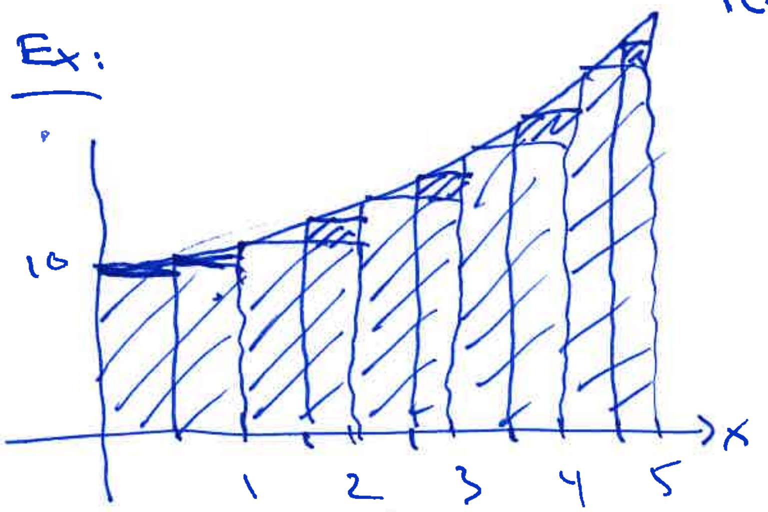


Total rent:

$10+12+14+16+18$

$= \frac{10+18}{2} \cdot 5 = \underline{\underline{70}}$

Ex:



$f(x) = 10 \cdot e^{0.2x}$

Total rent = area under the graph.

Approximation:

$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$

$f(0) = 10$, $f(1) = 10e^{0.2}$, $f(2) = 10e^{0.4}$, $f(3) = 10e^{0.6}$, $f(4) = 10e^{0.8}$

$= 10 + 10e^{0.2} + 10e^{0.4} + 10e^{0.6} + 10e^{0.8} = 10\,\frac{e-1}{e^{0.2}-1} \simeq \underline{77.6}$

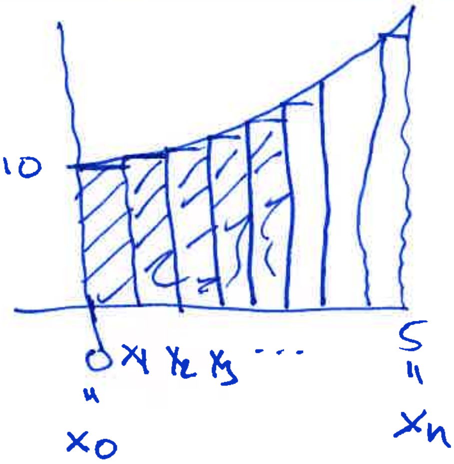
$\underline{n=10}$:

$10\cdot\frac{1}{2} + 10e^{0.1}\cdot\frac{1}{2} + 10e^{0.2}\cdot\frac{1}{2} + \dots + 10\cdot e^{0.9}\cdot\frac{1}{2}$

$= \frac{1}{2}\cdot 10\,\frac{e-1}{e^{0.1}-1} \simeq \underline{81.7}$

Riemann-sum:

$$f(x_0)\cdot\frac{5}{n} + f(x_1)\cdot\frac{5}{n} + \dots + f(x_{n-1})\cdot\frac{5}{n}$$



Definition: Definite integral

$$\int_0^5 e^{0.2x}\,dx = \text{area under the graph of } f(x)=e^{0.2x} \text{ in the interval } [0,5]$$

$$= \lim_{n\to\infty}\left(f(x_0)\cdot\frac{5}{n} + f(x_1)\cdot\frac{5}{n} + \dots + f(x_{n-1})\cdot\frac{5}{n}\right)$$

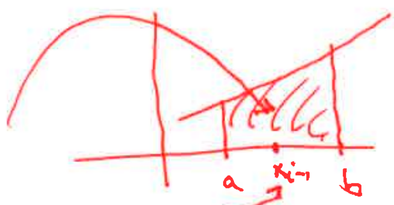
n subintervals:

$$\left.\begin{array}{l} [x_0,x_1] \\ [x_1,x_2] \\ [x_2,x_3] \\ \vdots \\ [x_{n-1},x_n] \end{array}\right\} n \text{ subintervals of length } 5/n$$

Defn:

Let $f(x)$ be a continuous function on $[a,b]$ with $f(x)\geq 0$ for x in $[a,b]$. Then

$$\int_a^b f(x)\,dx = \left\{\begin{array}{l}\text{area under the graph of } f \\ \text{on the interval } [a,b]\end{array}\right\}$$



$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_{i-1})\cdot\frac{b-a}{n}\right), \text{ where } x_{i-1} = a + (i-1)\cdot\frac{b-a}{n}$$

2 Antiderivatives and indefinite integrals

Ex: $f(x) = 2x$

An antiderivative of $f(x)$: A function $F(x)$ such that

$$F'(x) = 2x$$

$F(x) = x^2$ is an antiderivative:

$$(x^2)' = 2x$$

$F(x) = x^2 + 1$ — " —

$$(x^2+1)' = 2x$$

The general antiderivative of $f(x) = 2x$:

$F(x) = x^2 + C$, where C is a constant

Why is this the general antiderivative?

Assume that $F(x)$ is an antiderivative.

Then $F'(x) = 2x$

$$(x^2)' = 2x$$

$$\Downarrow$$

$$(F(x) - x^2)' = F'(x) - 2x = 2x - 2x = 0$$

$$\Downarrow$$

$F(x) - x^2 = C$, i.e. $F(x) = x^2 + C$

The indefinite integral:

$$\int 2x\,dx = \underline{\underline{x^2+C}}$$

Defn: The indefinite integral $\int f(x)\,dx$ is the general antiderivative of $f(x)$.

Result: If $F'(x) = f(x)$, then $\underline{\int f(x)\,dx = F(x)+C}$

dx : x is the integration variable

$\int$: integration sign

Notation: Indefinite integrals

$$\underset{\substack{\uparrow\\\text{integration}\\\text{sign}}}{\int} \underline{f(x)}\ \underset{\substack{\uparrow\\x\text{ is the}\\\text{integration}\\\text{variable}}}{dx} = \underset{\substack{\uparrow\\\text{an}\\\text{antiderivative}\\\text{of } f(x)\text{, i.e.}\\F'(x)=f(x)}}{F(x)} + \underset{\substack{\uparrow\\\text{constant}\\\text{of}\\\text{integration}}}{C}$$

integrand, the function we want to find antiderivatives to

3 Integration rules:

$$(x^n)' = n x^{n-1} \qquad \int n x^{n-1}\,dx = x^n + C$$

Integration rule:

$$\boxed{\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C}$$

Test:

$$\left(\frac{1}{n+1}\cdot x^{n+1}\right)' = \frac{1}{\cancel{n+1}}\cdot \cancel{(n+1)}\cdot x^n = x^n$$

ok

Ex:

$$\int x^2+1\,dx = \frac{1}{3}x^3 + x + C$$

$$\int x - x^4\,dx = \frac{1}{2}x^2 - \frac{1}{5}x^5 + C$$

$$\int 2x^2-3x\,dx = \int 2x^2\,dx - \int 3x\,dx$$
$$= 2\cdot\frac{1}{3}x^3 - 3\cdot\frac{1}{2}x^2 + C$$

Integration rules:

1. $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C \quad , \; n \neq -1$ (power rule)
2. $\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$
4. $\int c\cdot u\,dx = c\cdot \int u\,dx$

$\left.\right\}$ u, v : expressions in x; c : constant

5. $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}\cdot a^x + C \qquad (a>0)$

Ex:

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{3/2}\cdot x^{3/2} + C$$

$$(n = 1/2 \Rightarrow n+1 = 1/2 + 1 = 3/2)$$

$$= \frac{2}{3}x^{3/2} + C = \frac{2}{3}x\sqrt{x} + C$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx \qquad \begin{matrix} n=-2 \\ n+1=-1 \end{matrix}$$

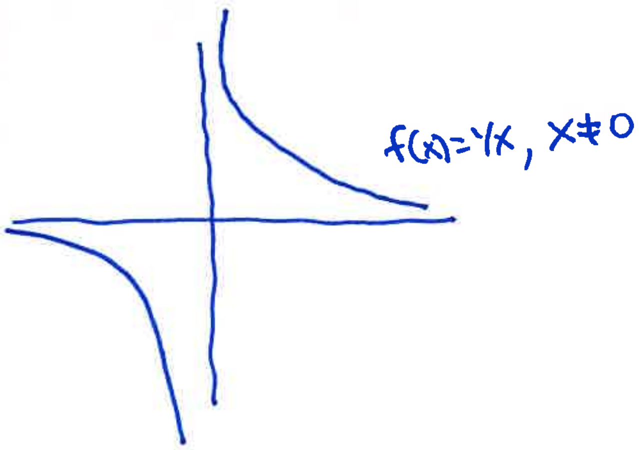
$$= \frac{1}{-1}\cdot x^{-1} + C$$

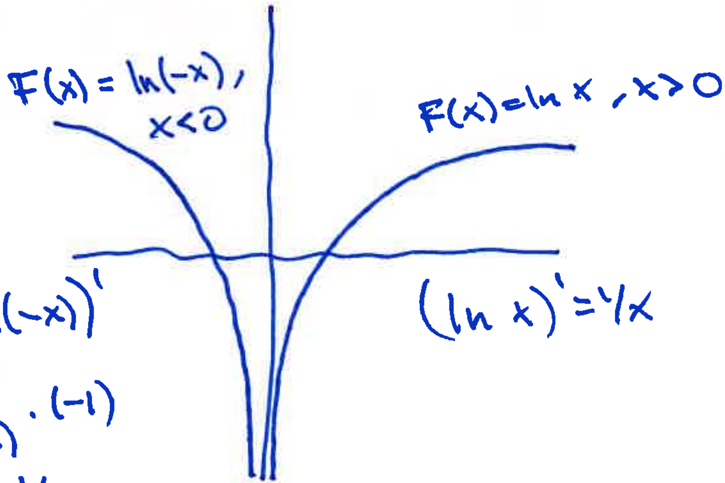
$$= -\frac{1}{x} + C$$

$$\int \frac{x^2+1}{x}\,dx = \int \frac{x^2}{x} + \frac{1}{x}\,dx$$

$$= \int x + 1/x\,dx = \frac{1}{2}x^2 + \ln|x| + C$$

Explanation: $\int \frac{1}{x}\,dx = \ln|x| + C$





$$(\ln(-x))' = \frac{1}{(-x)}\cdot(-1) = 1/x$$

$$F(x) = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \end{cases} = \ln|x|$$

Advanced integration techniques:

i) Substitution
ii) Integration by parts
iii) Partial fractions and integration of rational functions.

Continue next time

Substitution:

Ex: $\int e^{-x}\,dx = \frac{1}{(-1)} \cdot e^{-x} + C = \underline{-e^{-x} + C}$

$$\int e^{2x-1}\,dx = \frac{1}{2} \cdot e^{2x-1} + C$$

$$\left(\tfrac{1}{2} e^{2x-1}\right)' = \frac{1}{2} \cdot \left(e^{2x-1}\right)' = \frac{1}{\cancel{2}}\left(e^u \cdot \cancel{2}\right) = \underline{e^{2x-1}}$$

e^u, $u = 2x-1$

$$\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b} + C \qquad (a, b \text{ const. } a \neq 0)$$

$$\int (2x-1)^7\,dx = \frac{1}{2} \cdot \frac{1}{8} \cdot (2x-1)^8 + C = \frac{1}{16}(2x-1)^8$$

u^7, $u = 2x-1$, $\underline{u' = 2}$

$$\int e^{-x^2}\,dx \neq \frac{1}{-2x} \cdot e^u + C$$

"e^u", $u = -x^2$, $u' = -2x$

Formalism:

Ex: $\int e^{-x}\,dx = \int e^{u}\cdot \frac{du}{-1} = \int e^{u}\cdot\frac{1}{-1}\cdot du$

$\boxed{\begin{array}{l} u=-x \\ du = u'\cdot dx \end{array}}$

$\boxed{\begin{array}{l} u=-x \\ du = -\,dx \end{array}}$

$du = -1\cdot dx$

$\Downarrow$

$dx = \frac{du}{-1}$

$= -\int e^{u}du$

$= -e^{u}+C$

$= \underline{\underline{-e^{-x}+C}}$

$\int \underline{x\cdot e^{-x^2}}\,\underline{dx} = \int \cancel{x}\cdot e^{u}\cdot\frac{du}{-2\cancel{x}}$

$\boxed{\begin{array}{l} u=-x^2 \\ du = -2x\,dx \end{array}}$ $\quad dx = \frac{du}{-2x}$

$= \int e^{u}\cdot(-1/2)\cdot du$

$= -\frac{1}{2}\int e^{u}du = -\frac{1}{2}e^{u}+C$

$= \underline{\underline{-\frac{1}{2}e^{-x^2}+C}}$