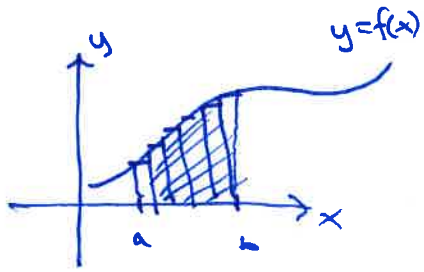
Plan

1. Substitution
2. Integration by parts
3. Integration of rational expressions and partial fractions

Review:

$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = \text{area under the graph of } f \text{ in } [a,b] \text{ (in case } f(x)\geq 0)$$



$$= \lim_{n\to\infty} \underbrace{f(x_1)\cdot\frac{b-a}{n} + f(x_2)\cdot\frac{b-a}{n} + \dots}_{\text{Riemann sum}}$$

$$\underbrace{\int f(x)\,dx}_{\text{indefinite integral}} = \left\{ \begin{array}{l}\text{the general}\\ \text{antiderivative}\\ \text{of } f(x)\end{array}\right\} = F(x) + C$$

if $F'(x) = f(x)$ (antiderivative of f)

Integration rules:

i) $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C$ $(n \neq -1)$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int u \pm v\,dx = \int u\,dx \pm \int v\,dx$

iv) $\int c\cdot u\,dx = c\cdot\int u\,dx$

u, v : any expr.
c : a constant

v) $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)}\cdot a^x + C$ $(a>0)$

Ex: $\int \frac{x}{1-x}\,dx$

i) Polynomial division:

$$\begin{array}{l} x : (-x+1) = -1 \\ \underline{-(x-1)} \\ \quad 1 \end{array} \quad \Bigg\} \quad \frac{x}{1-x} = -1 + \frac{1}{1-x}$$

$$\int \frac{x}{1-x}\,dx = \int -1 + \frac{1}{1-x}\,dx = -x + \int \frac{1}{1-x}\,dx$$

$$= \underline{\underline{-x - \ln|1-x| + C}}$$

$$\left(\int \frac{1}{1-x}\,dx = \int \frac{1}{u}\,\frac{du}{-1} = -\int \frac{1}{u}\,du = -\ln|u| + C \right.$$

$$\boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}} \quad dx = \frac{du}{-1} \qquad = \underline{-\ln|1-x| + C}$$

$$\left. \text{Substitution: } \boxed{du = u' \cdot dx} \right)$$

ii) Substitution:

$$\int \frac{x}{1-x}\,dx = \int \frac{x}{u}\,\frac{du}{-1} = \int \frac{1-u}{-u}\,du$$

$$\boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}} \rightarrow \begin{array}{l} u = 1-x \\ x = 1-u \end{array}$$

$$= \int \frac{1}{-u} + \frac{-u}{-u}\,du = \int -\frac{1}{u} + 1\,du$$

$$= -\ln|u| + u + C = \underline{\underline{-\ln|1-x| + 1 - x + C}}$$

1. Substitution: $\int f(x)\,dx \rightsquigarrow \int g(u)\,du$

$$\boxed{\begin{array}{l} u = \ldots \text{ (expr. in } x) \\ du = u' \cdot dx \end{array}}$$

$u' = \frac{du}{dx}$

We must choose u s.t. the new integral is easier.

You can try:

- u is the kernel of a composite fn.
- u is the denominator in a fraction

Ex: $\int x \cdot \ln(x^2+1)\,dx = \int \cancel{x} \cdot \ln(u) \cdot \frac{du}{2\cancel{x}}$

$$\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x \cdot dx \end{array}}$$

$= \frac{1}{2} \int \ln(u)\,du$ (to be continued)

$\boxed{\begin{array}{l}\int \ln x\,dx \\ = x\ln x - x + C\end{array}}$

$= \frac{1}{2}\left(u \cdot \ln(u) - u\right) + C$

$v = u \qquad w = \ln u$

$v' = 1 \qquad w' = 1/u$

or

$= \frac{1}{2}\left(\int \underset{w}{\ln u} \cdot \underset{v'}{1}\,du \right)$

$= \frac{1}{2}\left(u \cdot \ln(u) - \int u \cdot 1/u\,du \right)$

$= \frac{1}{2}\left(u \cdot \ln(u) - \int 1\,du \right)$

$= \frac{1}{2}\left(u \cdot \ln(u) - u \right) + C$

$= \frac{1}{2}\left[(x^2+1)\ln(x^2+1) - (x^2+1) \right] + C$

Ex: $\int x\cdot\sqrt{x^2+1}\,dx = \int \cancel{x}\cdot\sqrt{u}\cdot\frac{du}{2\cancel{x}}$

$$\boxed{\begin{array}{c} u=x^2+1 \\ du=2x\cdot dx \end{array}}$$

$$= \int \tfrac{1}{2}\cdot\sqrt{u}\,du = \tfrac{1}{2}\int u^{1/2}\,du = \tfrac{1}{2}\left(\frac{1}{3/2}\cdot u^{3/2}\right)+C$$

$$= \frac{1}{3}u^{3/2}+C = \tfrac{1}{3}u\sqrt{u}+C$$

$$= \underline{\underline{\tfrac{1}{3}(x^2+1)^{3/2}+C}} = \underline{\underline{\tfrac{1}{3}(x^2+1)\sqrt{x^2+1}+C}}$$

2) Integration by parts:

Ex: $\int x\cdot e^x\,dx \quad \boxed{\neq \tfrac{1}{2}x^2\cdot e^x + C}$

$\cancel{\int x\cdot e^x\,dx = \tfrac{1}{2}x^2\cdot e^x - \int \tfrac{1}{2}x^2\cdot e^x\,dx}$

$\cancel{\boxed{\begin{array}{ll} u=\tfrac{1}{2}x^2 & v=e^x \\ u'=x & v'=e^x \end{array}}}$

$\int x\cdot e^x\,dx = xe^x - \int e^x\cdot 1\,dx$

$$\boxed{\begin{array}{ll} u=e^x & v=x \\ u'=e^x & v'=1 \end{array}}$$

$= xe^x - \int e^x dx = \underline{\underline{xe^x - e^x + C}}$

Remember:

$(u\cdot v)' = u'\cdot v + u\cdot v'$

$\int (u\cdot v)'\,dx = \int u'\cdot v\,dx + \int u\cdot v'\,dx$

$uv = \int u'v\,dx + \int u\cdot v'\,dx$

$$\boxed{\int u'\cdot v\,dx = u\cdot v - \int u\cdot v'\,dx}$$

formula for integration by parts

Integration by parts: $\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx}$

is used to integrate products

Ex: $\int \ln x \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx = x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$

$= x \cdot \ln x - \int 1 \, dx$

$= x \cdot \ln x - x + C$

$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$

$\boxed{\int \ln x \, dx = x \ln x - x + C}$ formula for $\int \ln x \, dx$

$(x \ln x - x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$

$= \ln x + 1 - 1 = \ln x$

Ex: $\int \frac{\ln x}{x} \, dx = \int \frac{\ln x}{x} \, dx = \int \underbrace{\ln x}_{v} \cdot \underbrace{\frac{1}{x}}_{u'} \, dx$

$= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x} \, dx$

$\boxed{\begin{array}{ll} u = \ln x & v = \ln x \\ u' = 1/x & v' = 1/x \end{array}}$

$I = \int \ln x \cdot \frac{1}{x} \, dx$: $I = (\ln x)^2 - I$

$2I = (\ln \cancel{x})^2 + C$

$I = \frac{1}{2} (\ln x)^2 + c$

Alt:

$\int \frac{\ln x}{x} \, dx = \int \frac{u}{x} \cdot \frac{du}{1/x} = \int \frac{u}{1} \, du = \int u \, du$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} \cdot dx \end{array}}$

$= \frac{1}{2} u^2 + C = \frac{1}{2} (\ln x)^2 + C$

③ Integration of rational expressions:

Rational expression: $\frac{p(x)}{q(x)}$ where p, q are polynomials

Ex: $\int \frac{2}{1-x}\,dx = \int \frac{2}{u}\cdot\frac{du}{-1}$

$u = 1-x$
$du = -1\cdot dx$

works when

$\deg p = 0$
$\deg q = 1$

$= -2\int \frac{1}{u}\,du = -2\ln|u| + C = -2\ln|1-x| + C$

$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\cdot\frac{du}{a}$

$u = ax+b$
$du = a\,dx$

$= \frac{A}{a}\cdot\int \frac{1}{u}\,du$

$= \frac{A}{a}\cdot\ln|u| + C = \frac{A}{a}\cdot\ln|ax+b| + C$

Formula: $$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C}$$ $(a \neq 0)$

What if:

① $\deg q(x) \geq 2$

② $\deg p(x) \geq \deg q(x)$

What if $\deg p(x) \geq \deg q(x)$ in $\int \frac{p(x)}{q(x)}\,dx$?

Ex: $\int \frac{x^2}{x-1}\,dx \rightarrow$ polynomial division

$$\left.\begin{array}{l} x^2 : (x-1) = x+1 \\ -(x^2-x) \\ \hline \quad x \\ \quad -(x-1) \\ \hline \qquad 1 \end{array}\right\} \quad \frac{x^2}{x-1} = \underbrace{x+1}_{\text{poly.}} + \frac{1}{x-1}$$

$$\int \frac{x^2}{x-1}\,dx = \int x+1 + \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}x^2 + x + \frac{1}{1}\ln|x-1| + C$$

$$= \underline{\underline{\frac{1}{2}x^2 + x + \ln|x-1| + C}}$$

What if deg $q(x) \geq 2$ in $\int \frac{p(x)}{q(x)}\,dx$

Ex: $\int \frac{2}{x^2-1}\,dx = \int \frac{2}{u} \cdot \frac{du}{2x}$ too hard

$\boxed{\begin{aligned} u &= x^2-1 \\ du &= 2x\,dx \end{aligned}}$

Partial fractions: Decomposition of fractions

$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

1) Factorize the denominator:

$$x^2-1 = (x-1)(x+1)$$

2) Try to find const. A and B such that

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Big| \cdot (x-1)(x+1)$$

$$2 = \frac{A\,\cancel{(x-1)}(x+1)}{\cancel{x-1}} + \frac{B(x-1)\cancel{(x+1)}}{\cancel{x+1}}$$

$$2 = \underset{\underset{1}{\shortparallel}}{A} \cdot (x+1) + \underset{\underset{-1}{\shortparallel}}{B}\,(x-1)$$

Alt. 1:

$x=-1$: $2 = A \cdot 0 + B \cdot (-2)$

$2 = -2B \Rightarrow \underline{B=-1}$

$x=1$: $2 = A \cdot 2 + B \cdot 0$

$2 = 2A \Rightarrow A = 1$

Alt 2: $2 = A(x+1) + B(x-1)$

$= Ax + A + Bx - B$

$0 \cdot x + 2 = (A+B)x + (A-B)$

$$\begin{aligned} A+B &= 0 \\ A-B &= 2 \\ \hline 2A &= 2 \end{aligned} \qquad \underline{A=1} \quad \underline{B=-1}$$

Ex:

$$\int \frac{2}{x^2-1}\,dx = \int \frac{2}{(x-1)(x+1)}\,dx = \int \frac{A}{x-1} + \frac{B}{x+1}\,dx$$

$$= \int \frac{1}{x-1} + \frac{-1}{x+1}\,dx$$

$$= \ln|x-1| - \ln|x+1| + C$$