Plan

1 Definite integrals and antiderivatives
2 Computing areas between graphs
3 Improper integrals

Problem 10, Problem Set 18

No time today, will go through it next week

Review:

Integration techniques:

i) Substitution
ii) Int. by parts
iii) Partial fraction

i) $\int f(x)\,dx = \int g(u)\cdot du$

$\boxed{\begin{array}{c} u = u(x) \\ du = u'(x)\cdot dx \end{array}}$

ii) $\int u'v\,dx = uv - \int uv'\,dx$

$\boxed{\begin{array}{ll} u = & v = \cdot \\ u' = \cdot & v' = \end{array}}$

iii) $\int \frac{p(x)}{q(x)}\,dx$ ← if deg $p \geq$ deg q, use polynomial division first

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \int \frac{U}{(x-x_1)} + \frac{V}{(x-x_2)}\,dx$$

Partial fractions: $\frac{Ax+B}{ax^2+bx+c} = \frac{U}{x-x_1} + \frac{V}{x-x_2}$

Ex: $\frac{2x}{x^2-5x+6} = \frac{U}{x-2} + \frac{V}{x-3} \quad \Big| \cdot (x-2)(x-3)$

$\boxed{x^2-5x+6 = (x-2)(x-3)}$

$2x = U\cdot(x-3) + V(x-2)$

$2x = Ux + Vx + (-3U-2V) = (U+V)x + (-3U-2V)$

$U+V=2 \qquad V = 2-U \qquad V = 2-(-4) = 6$

$-3U-2V=0 \qquad -3U-2(2-U)=0$

$-3U-4+2U=0$

$-U-4=0$

$U=-4$

$\int \frac{2x}{x^2-5x+6}\,dx = \int \frac{-4}{x-2} + \frac{6}{x-3}\,dx$

$= -4\ln|x-2| + 6\ln|x-3| + C$

Ex: $\int \frac{2}{x^2-4x+4}\,dx = \int \frac{A}{x-2} + \frac{B}{(x-2)^2}\,dx$

$\boxed{x^2-4x+4 = (x-2)^2}$

$\frac{2}{x^2-4x+4} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} \quad \Big| \cdot (x-2)^2$

$2 = A(x-2) + B$

$0\cdot x + 2 = Ax + (-2A+B)$

$A=0 \qquad -2A+B=2$

$B=2$

$\int \frac{2}{x^2-4x+4}\,dx = \int \cancel{\frac{0}{x-2}} + \frac{2}{(x-2)^2}\,dx = \int \frac{2}{u^2}\,du = \int 2u^{-2}\,du$

$\boxed{u=x-2 \quad du=dx}$

$= 2\cdot\frac{1}{-1}u^{-1} + C$

$= -2\cdot\frac{1}{x-2} + C$

Ex: $\int \frac{1}{x^2+1}\,dx = \arctan(x)+C = \tan^{-1}(x)+C$

Pb. Sheet 18:

6b) $\int x\cdot\ln(1-x)\,dx = \frac{1}{2}x^2\cdot\ln(1-x) - \int \frac{1}{2}x^2\cdot\frac{-1}{1-x}\,dx$

$u=\frac{1}{2}x^2 \qquad v=\ln(1-x)$
$u'=x \qquad v'=\frac{-1}{1-x}$

$= \frac{1}{2}x^2\ln(1-x) + \frac{1}{2}\int\frac{x^2}{1-x}\,dx$

$x^2 : 1-x = -x-1$
$-(x^2-x)$
x
$-(x-1)$
1

$= \frac{1}{2}x^2\ln(1-x) + \frac{1}{2}\int -x-1+\frac{1}{1-x}\,dx$

$= \frac{1}{2}x^2\ln(1-x) - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2}\ln(1-x)+C$

6c) $\int\frac{x^3+x^2-2x-6}{x^2-1}\,dx = \int x+1+\frac{-x-5}{x^2-1}\,dx$

$x^3+x^2-2x-6 : x^2-1 = x+1$
$-(x^3 \quad -x \quad)$
x^2-x-6
$-(x^2 \quad -1)$
$-x-5$

$= \frac{1}{2}x^2+x+\int\frac{A}{x-1}+\frac{B}{x+1}\,dx$

$= \frac{1}{2}x^2+x+\int\frac{-3}{x-1}\,dx+\int\frac{2}{x+1}\,dx$

$= \frac{1}{2}x^2+x-3\ln|x-1|+2\ln|x+1|+C$

$\frac{-x-5}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1} \quad |\cdot(x-1)(x+1)$

$-x-5 = A(x+1)+B(x-1) = (A+B)x+(A-B)$

$A+B=-1$
$A-B=-5$
$2A=-6$
$A=-3$
$B=2$

7) $$\int 2x^3 e^{-x^2}\,dx = \int 2x^3 e^u \cdot \frac{du}{-2x}$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$= \int -x^2 e^u\,du = \int u e^u\,du$$

$$\boxed{\begin{aligned} v &= e^u \qquad & w &= u \\ v' &= e^u \qquad & w' &= 1 \end{aligned}}$$

$$= ue^u - \int e^u \cdot 1\,du$$

$$= ue^u - \int e^u\,du = ue^u - e^u + C$$

$$= -x^2 e^{-x^2} - e^{-x^2} + C$$

10) No time today; will do it next time

① Definite integrals and antiderivatives

Defn I: $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f

General defn. of a definite integral

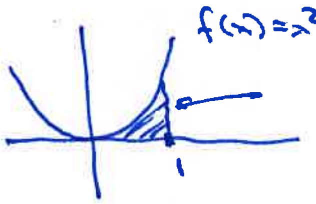
the answer/result is a number.

Defn 2: If $f(x) \geq 0$ is a cont. fn. on $[a,b]$, then $\int_a^b f(x)\,dx$ is the area under the graph of f in $[a,b]$.

Using Defn I

Ex: $\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3 + \cancel{C}\right]_0^1 = \left(\tfrac{1}{3}\cdot 1^3 + \cancel{C}\right) - \left(\tfrac{1}{3}\cdot 0^3 + \cancel{C}\right)$

$\int x^2\,dx = \tfrac{1}{3}x^3 + C$ $\qquad = \tfrac{1}{3} - 0 = \underline{\underline{1/3}}$



Area $= \int_0^1 x^2\,dx = 1/3$

by Defn. 2

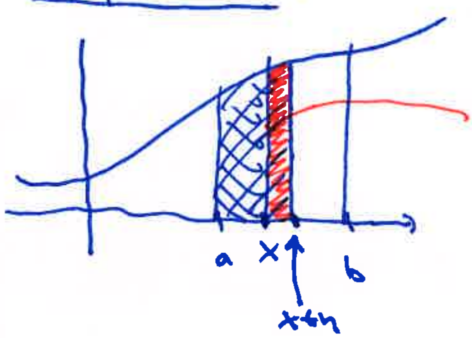
Theorem:

If $f(x)$ is a continuous function on $[a,b]$ and $f(x) \geq 0$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is the area under the graph of $f(x)$ in $[a,b]$.

Explanation:



Define an area function:

$A(x)$ = area under $f(x)$ in $[a,x]$

$A(b)$ = area on all of $[a,b]$

$A(a) = 0$

Claim. $A'(x) = f(x)$

$$\frac{A(x+h) - A(x)}{h} = \frac{\square - \square}{h} \simeq \frac{f(x)\cdot \cancel{h}}{\cancel{h}} = f(x)$$

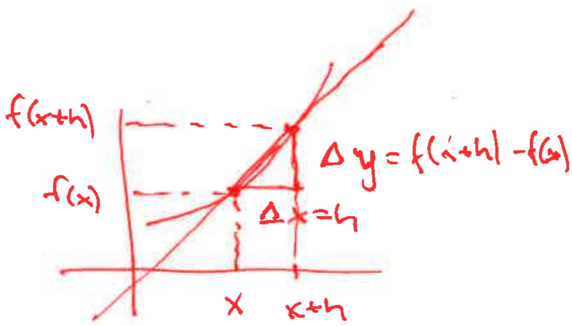
This means: $A(x)$ is an antiderivative of $f(x)$

$\Downarrow$

$$\int_a^b f(x)\,dx = F(b) - F(a) = A(b) - A(a) = A(b) - 0$$
$$= A(b) = \text{area under } f(x) \text{ on } [a,b].$$

Remember: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

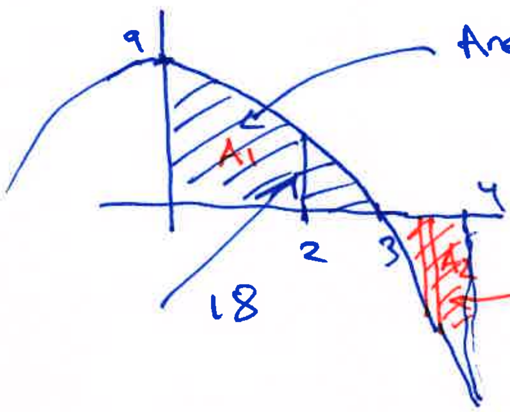
Defn. of the derivative of any fn. $f(x)$



Ex:

$$\int_0^2 9-x^2\,dx = \left[9x-\tfrac{1}{3}x^3\right]_0^2$$

$$= \left(9\cdot 2-\tfrac{1}{3}\cdot 2^3\right)-\left(0\right) = 18-\frac{8}{3} = 15\tfrac{1}{3} = \underline{\underline{46/3}}$$



Area $= 46/3 \approx \underline{15.33}$

Area $= \int_3^4 (0-f(x))\,dx$

$= \int_3^4 -f(x)\,dx = -\int_3^4 f(x)\,dx$

Ex:

$$\int_0^4 9-x^2\,dx = \left[9x-\tfrac{1}{3}x^3\right]_0^4$$

$$= \left(9\cdot 4-4^3/3\right)-0 = 36-\frac{64}{3} \approx \underline{14{,}67}$$

$$\int_0^4 9-x^2\,dx = \underbrace{\int_0^3 9-x^2\,dx} + \underbrace{\int_3^4 9-x^2\,dx}$$

$$= \left[9x-\tfrac{1}{3}x^3\right]_0^3 + \left[9x-\tfrac{1}{3}x^3\right]_3^4$$

$$= \left(9\cdot 3-\frac{27}{3}\right)-0 + \left(9\cdot 4-\frac{64}{3}\right)-\left(27-\frac{27}{3}\right)$$

$$= (27-9) + \left(36-64/3\right)-\left(27-27/3\right)$$

$$= 18 + 9-37/3$$

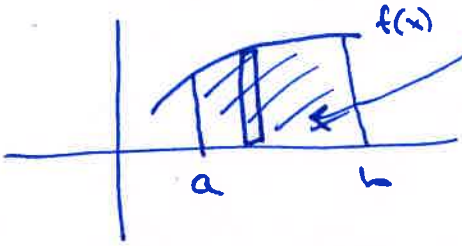
$$= 18 + (-10/3) \approx 14{,}67$$

$$\approx 18 - 3.33$$

$$A_1 - A_2$$

(2) Computing areas between graphs

i) When $f(x) \geq 0$ on $[a,b]$: $A = \int_a^b f(x)\,dx$

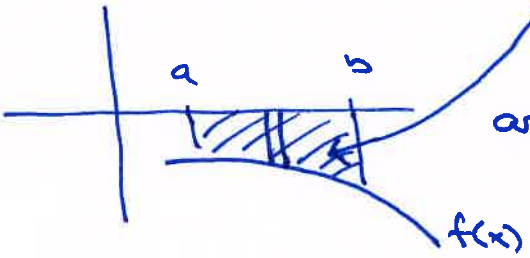


area under the graph of f in $[a,b]$

ii) When $f(x) \leq 0$ on $[a,b]$: $A = \int_a^b -f(x)\,dx = -\int_a^b f(x)dx$

$$\Updownarrow$$

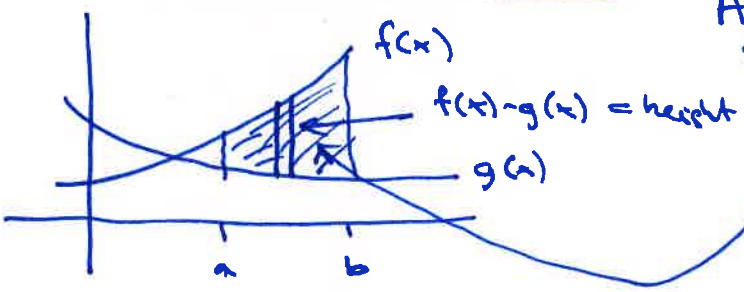
$$\int_a^b f(x)\,dx = -A$$



area over the graph of f in $[a,b]$ (between $y=f(x)$ and $y=0$)

iii) When $f(x) \geq g(x)$ on $[a,b]$:

$$A = \int_a^b f(x) - g(x) \quad dx$$



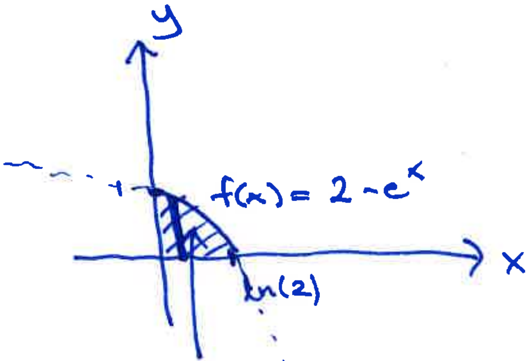
area between $y=f(x)$ and $y=g(x)$ on $[a,b]$

Remember:

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

Ex: R : area bounded by x-axis, y-axis, and $y = 2 - e^x$

Find the area of R: A



$x=0$: $y = 2 - e^0 = 2 - 1 = 1$

$y=0$: $0 = 2 - e^x$

$e^x = 2$

$x = \ln(2)$

$$A = \int_0^{\ln(2)} f(x)\,dx = \int_0^{\ln(2)} 2 - e^x\,dx$$

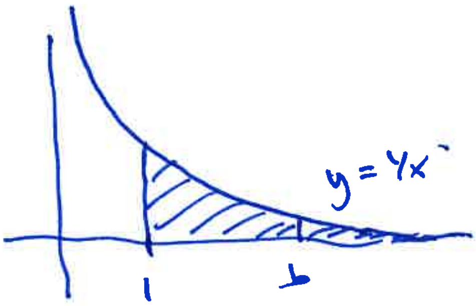
$$= \Big[2x - e^x\Big]_0^{\ln(2)} = \left(2\ln 2 - e^{\ln 2}\right) - \left(2\cdot 0 - e^0\right)$$

$$= 2\ln(2) - 2 + 1 = 2\ln 2 - 1 \approx 0.4$$

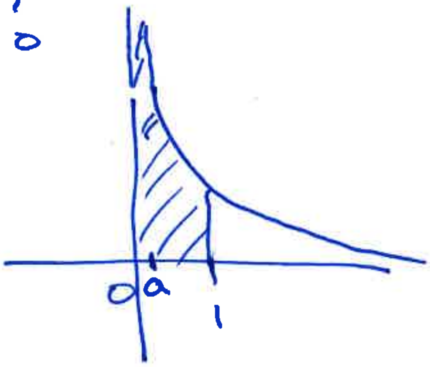
3 Improper integrals:

$$\int_1^\infty \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^\infty = \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} \left(\ln b - \ln 1\right) = \lim_{b\to\infty} \ln b = \infty$$

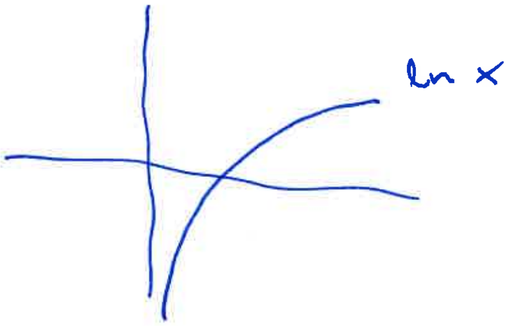


$$\int_0^1 \frac{1}{x}\,dx = \left[\ln|x|\right]_0^1 = \lim_{a\to 0^+}\left[\ln|x|\right]_a^1$$



$$= \lim_{a\to 0^+}\left(\ln 1 - \ln a\right)$$

$$= \lim_{a\to 0^+}\left(-\ln a\right) = \underline{\underline{\infty}}$$



$\int_a^b f(x)\,dx$ is improper if $\begin{cases} \text{i) } b=\infty \text{ or } a=-\infty \\ \text{ii) } f \text{ not defined at } x=a \text{ or } x=b \end{cases}$