EBA 2911, Lecture 2, 23 Aug 2019, Runar Ile

Plan
1. Repetion & some of the exercises
2. Relative change and rate of change
3. Powers
4. Interest
5. Present value of cash flow

1. Repetition.

Algebraic expression: $3ab^2 - 7$, $2x^3 - 11x + 15$

Laws of algebra: $a(b+c) = ab + ac$ (distr. law) ...

Roots: Square root $\sqrt{b}$ is only defined if $b \geq 0$ and then equals the number $a \geq 0$ s.t. $a^2 = b$

Ex: $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} = |x|$

The third root $\sqrt[3]{b}$ is defined for all b and it is the number a s.t. $a^3 = b$

Ex: $\sqrt[3]{-8} = -2$

Powers: $(-2)^3 = (-2)\cdot(-2)\cdot(-2) = -8$

$$\frac{4^3}{2^5} = \frac{4\cdot 4\cdot 4}{2\cdot 2\cdot 2\cdot 2\cdot 2} = \frac{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 2\cdot 2\cdot 2} = \frac{2}{1} = 2$$

$$= \frac{(2^2)^3}{2^5} = \frac{2^{6}}{2^5} = 2^{6-5} = 2^1 = 2.$$

$6 = 2\cdot 3$

Order of operations: $2+3\cdot 4 = 14$

$(2+3)\cdot 4 = 20$

$3\cdot 2^4 = 3\cdot 2\cdot 2\cdot 2\cdot 2 = 48$

$(3\cdot 2)^4 = 6^4 = 1296$

$-3^2 = (-1)\cdot 3\cdot 3 = (-1)\cdot 9 = -9$

Some problems from the exercise session

2i) $$\frac{x^2-3x}{x(y-3)}\cdot\frac{xy^2-9x}{x-3} = \frac{(x^2-3x)\cdot(xy^2-9x)}{x(y-3)\cdot(x-3)}$$

$$= \frac{\cancel{x}\cancel{(x-3)}\cdot x(y^2-9)}{\cancel{x}(y-3)\cancel{(x-3)}} = \frac{x(y^2-9)}{y-3} = \frac{x\cancel{(y-3)}(y+3)}{\cancel{y-3}}$$

$$= \underline{\underline{x(y+3)}}$$

3d) $$x+3+\frac{2}{x-1} = \frac{x+3}{1}+\frac{2}{x-1} = \frac{x+3}{1}\cdot\underbrace{\frac{x-1}{x-1}}_{=1}+\frac{2}{x-1}$$

$$= \frac{(x+3)(x-1)}{x-1}+\frac{2}{x-1} = \frac{(x+3)(x-1)+2}{x-1}$$

$$= \underline{\underline{\frac{x^2+2x-1}{x-1}}} = \underline{\underline{\frac{(x+1-\sqrt{2})(x+1+\sqrt{2})}{x-1}}}.$$

7e) $(x-4)^2 = 9 \iff x-4 = 3$ or $x-4 = -3$

$\underline{\underline{x = 7}}$ or $\underline{\underline{x = 1}}$

Check $(x=7)$: lhs is $(7-4)^2 = 3^2 = 9$

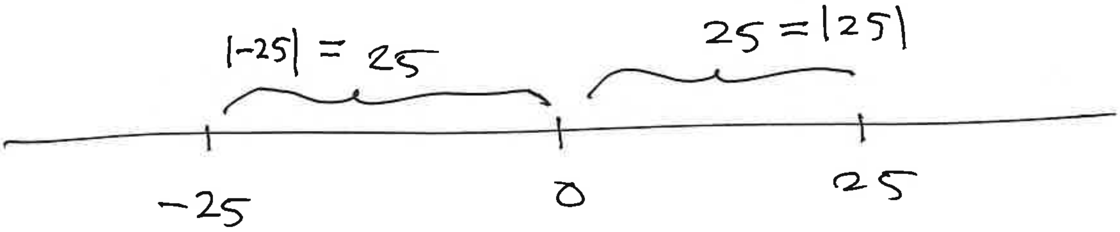
rhs is 9 — so ok.

$(x=1)$ lhs is $(1-4)^2 = (-3)^2 = 9$

rhs is 9 — so ok

7g) $|x| = 25$. The distance from x to 0 should be 25 so $\underline{\underline{x = 25}}$ or $\underline{\underline{x = -25}}$

$\underline{\underline{\text{Or}}}$: $|x| = 25 \iff \underline{\underline{x = 25}}$ or $-x = 25$ so $\underline{\underline{x = -25}}$



7k) $|x-2| = 25 \iff x-2 = 25$ or $x-2 = -25$

$x = \underline{\underline{27}}$ or $x = \underline{\underline{-23}}$

7j) $x|x| = 9$

Two cases: $\underline{\text{Either } x \geq 0}$: $x \cdot x = 9$ so $x = \pm 3$

but since $x \geq 0$ only $x = \underline{\underline{3}}$ is 'good'.

Or $\underline{x < 0}$: $x \cdot (-x) = 9 \iff x^2 = -9$ no solutions

8h) $x^3 = 1.03^{-12}$. We have $1.03^{-12} = \dfrac{1}{1.03^{12}}$

and so $x = \sqrt[3]{\dfrac{1}{1.03^{12}}} = \dfrac{\sqrt[3]{1}}{\sqrt[3]{1.03^{12}}} = \dfrac{1}{1.03^{4}}$ $\leftarrow \frac{12}{3}$

(since $(1.03^4)^3 = 1.03^4 \cdot 1.03^4 \cdot 1.03^4 = 1.03^{4+4+4}$)

$= \underline{1.03^{-4}} = 0,888$

Calculator: 1.03 $\boxed{y^x}$ 4 $\boxed{=}$ $\boxed{1/x}$

Homework: Calculate $2 + 3 \cdot 4$ on the calc.

2 $\boxed{+}$ 3 $\boxed{\times}$ 4 $\boxed{=}$

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2. Relative change and rate of change

$$\text{Relative change} = \frac{\text{new value} - \text{old value}}{\text{old value}}$$

Recall: $\% = \frac{1}{100} = 0{,}01$, $3\% = 3 \cdot \frac{1}{100} = \frac{3}{100} = 0{,}03$

Ex: Kåre's hourly wage has increased from 163 kr to 181 kr. Then the relative change in Kåre's hourly wage is $\dfrac{181 - 163}{163} = \dfrac{18}{163} = 11.0\%$

Rate of change = 1 + relative change

Ex: The rate of change in Kåre's hourly wage is $1 + 0.11 = 1.11$

Ex: Last year Kåre earned 54 000 (w. 163/hour)

If he works as much as last year with the new wage he would earn

$$54\,000 \cdot 1.11 = 59.940$$

3. Powers

$$1.11^3 = 1.11 \cdot 1.11 \cdot 1.11$$

$$1.11^{-3} = \frac{1}{1.11^3}$$

$$1.11^{\frac{2}{3}} = \sqrt[3]{1.11^2}$$

For integers m, n and $n > 0$ and $a \geq 0$

then $$a^{\frac{m}{n}} \overset{\text{definition}}{=} \sqrt[n]{a^m}$$

Problem: Calculate $1.11^{\sqrt{2}}$ on your calculator!

(Answer: 1.159035...)

Answer: 1.11 $\boxed{y^x}$ 2 $\boxed{\sqrt{x}}$ $\boxed{=}$

Same base: $2^{1.5} \cdot 2^{3.8} = 2^{1.5+3.8} = 2^{5.3}$

Same exponent: Ex: $2^4 \cdot 3^4 = 2\cdot2\cdot2\cdot2 \cdot 3\cdot3\cdot3\cdot3$

$= 2\cdot3 \cdot 2\cdot3 \cdot 2\cdot3 \cdot 2\cdot3$

$= (2\cdot3)^4 = 6^4$

Ex: $\sqrt{2}\cdot\sqrt{3} = 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = (2\cdot3)^{\frac{1}{2}} = \sqrt{6}$

Pattern: $a^r \cdot b^r = (ab)^r$

Problem: Calculate 2^{-1} on the calculator.

Solution 1: $2 \; \boxed{y^x} \; 1 \; \boxed{+/-} \; \boxed{=}$

Solution 2: $2 \; \boxed{1/x} \; \boxed{=}$

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4 Interest Compounded interest and growth factor.

Ex: You deposit 40 000 into an account earning 2.3% annual interest.
Interest is added after each year (yearly compounding of interest)
After a year the balance (what's in the account) is given as $40\,000 + 40000 \cdot 2.3\%$

$= 40000 \cdot 1 + 40000 \cdot 0.023 = 40000 \cdot \underbrace{(1 + 0.023)}_{\text{growth factor}}$

$= 40\,920.00$

The balance after 5 years is

$$40000 \cdot 1.023^5 = \underline{\underline{44\,816.52}}$$

Ex: You deposit 40 000 with 2.3% nominal annual interest, but with quarterly compounding of interest.

Then the interest rate per period is $\frac{2.3\%}{4} = 0.575\%$

The growth factor for one period (=3 months) is $1 + 0.575\% = 1.00575$

After 1 year the balance is

$$40000 \cdot 1.00575^4 = \underline{\underline{40927.97}}$$

The effective annual interest is given by the annual growth factor which is

$$1.00575^4 = 1.023199$$

so effective interest is 0.023199 $= 2.3199\%$

nominal interest

Pattern: $B = B_0 \cdot \left(1 + \frac{r}{n}\right)^{m}$ — number of periods

balance after m periods

deposit (principal)

interest periods per year

Effective interest $r_{eff} = \underbrace{\left(1+\frac{r}{n}\right)^n}_{\text{growth factor for 1 year}} - 1$

5. Present value of cash flow.

Let K_0 be some investment/deposit/payment today. The <u>future value</u> K_n of K_0 in n years (or more generally n periods) with interest r is

$$K_n = K_0 \cdot (1+r)^n \qquad (*)$$

The opposite: Suppose K_n will be paid in n years (periods) with period interest r. Then the <u>present value</u> K_0 of K_n is given as

$$K_0 = \frac{K_n}{(1+r)^n}$$

(solve the equation (*) for K_0)

Ex: 30 mill. is paid 5 years from now with 8% (annual) interest has present value

$$K_0 = \frac{30 \text{ mill}}{1.08^5} = \underline{\underline{20.42 \text{ mill}}}$$

<u>Cash flow</u>

Ex: You pay 20 mill. today and will get paid back

6 mill after 3 years, 7 mill after 4 years and

8 mill after 5 years

The present value of cash flow

$$\underset{\text{now}}{(-20}, \underset{3y.}{6}, \underset{4y.}{7}, \underset{5y.}{8})$$

with 8% interest

is

$$-20 + \frac{6}{1.08^3} + \frac{7}{1.08^4} + \frac{8}{1.08^5}$$

$$= -4.65$$

The interest which makes the present value of cash flow equal to 0 is called the <u>internal rate of return</u> („internrenten")

<u>Homework</u>: Determine the internal rate of return for the <u>ex</u> above. (Answer: 1.1197%)