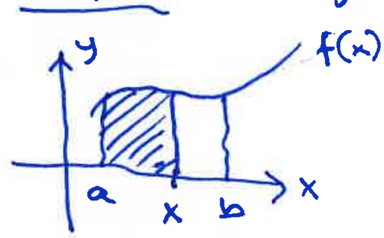
Plan

1. Economic applications of integration
2. Summary: Integration

Review:

i) 

$A(x)$ = area bounded by $a \le x$, $0 \le y \le f(x)$

$= \int_a^x f(x)\,dx$

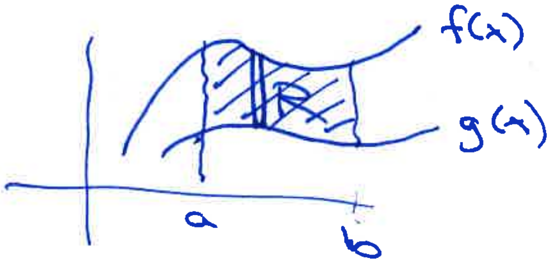
Important facts: i) $A'(x) = f(x)$, that is, $A(x)$ is an antiderivative of $f(x)$

ii) $A = \int_a^b f(x)\,dx = F(b) - F(a)$

for any antiderivative $F(x)$ of $f(x)$.

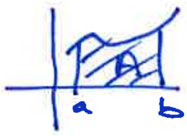
ii) If $f(x) \ge g(x)$ for $a \le x \le b$, then

$A(R) = \int_a^b f(x) - g(x)\,dx$



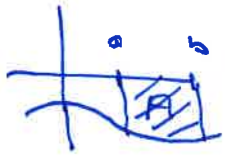
Special cases:

$f(x) \ge 0$ $\quad \int_a^b f(x)\,dx = A$



$\cancel{f(x) \le 0}$

$f(x) \le 0$ $\quad \begin{cases} \int_a^b -f(x)\,dx = A \\ \Updownarrow \\ \int_a^b f(x)\,dx = -A \end{cases}$



iii) Improper integrals:

$\int_a^b f(x)\,dx$ in case $\begin{cases} \text{i) } a \text{ or } b \text{ is } \pm\infty \\ \text{ii) } f(x) \to \pm\infty \text{ when } x \to a/b \end{cases}$ Solve using limits

Problem Set 18, Problem 10:

$$f(x) = 0.60\ln(1+x) + 0.40\ln(1-x), \quad 0 \leq x < 1$$

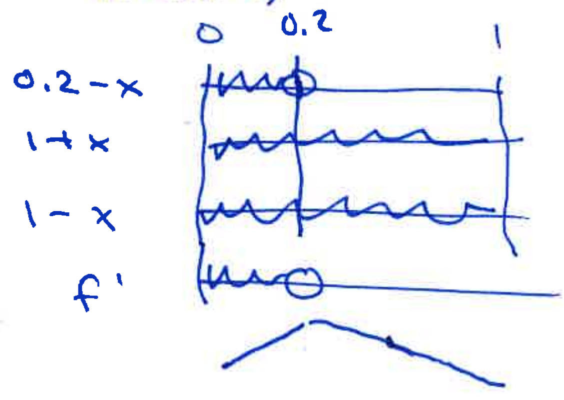
a) Find max. pt $x = x^*$ and max value $f(x^*)$.

$$f' = 0.6 \cdot \frac{1}{1+x} \cdot 1 + 0.40 \cdot \frac{1}{1-x} \cdot (-1)$$

$$= \frac{0.6}{1+x} - \frac{0.4}{1-x} = \frac{0.6(1-x)}{(1+x)(1-x)} - \frac{0.4(1+x)}{(1-x)(1+x)}$$

$$= \frac{0.6 - 0.6x - 0.4 - 0.4x}{(1+x)(1-x)} = \frac{0.2 - x}{(1+x)(1-x)}$$

$f'(x) = 0 \iff x = 0.2$



$x = 0.2$ is max pt.

$f(0.2) = 0.6\ln(1.2) + 0.4 \cdot \ln 0.8$

$\simeq 0.0201$ is max value

b) $$f''(x) = \left(\frac{0.6}{1+x} - \frac{0.4}{1-x}\right)' = \left(0.6(1+x)^{-1} - 0.4 \cdot (1-x)^{-1}\right)'$$

$$= 0.6 \cdot (-1)(1+x)^{-2} \cdot 1 - 0.4 \cdot (-1) \cdot (1-x)^{-2} \cdot (-1)$$

$$= \frac{-0.6}{(1+x)^2} - \frac{0.4}{(1-x)^2} < 0 \quad \text{for all } x \text{ in } [0,1)$$

Hence f is concave

c) Show that $f(x) < 0$ when $x > 2x^*$.

$x > 2x^* = 0.40$: $f(0.4) = 0.6\ln 1.4 + 0.4 \cdot \ln 0.6$

$\simeq -0.0024$

Since f is decreasing in $(0.4, 1)$ and $f(0.4) < 0$, it follows that $f(x) < 0$ for $x > 2x^* = 0.4$.

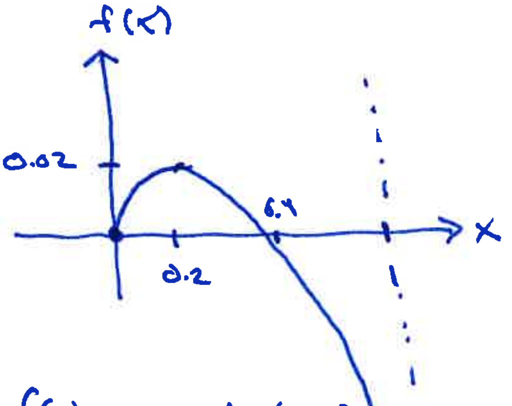
d) $\int_{-2}^{2} f(x)\,dx = \frac{18}{5} = A_1 - A_2$

$$\left(\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = A_1 - A_2 \right)$$

$$\frac{18}{5} = A_1 - A_2 = A_1 - \frac{22}{15}$$

$$A_1 = \frac{18}{5} + \frac{22}{15} = \frac{18 \cdot 3}{5 \cdot 3} + \frac{22}{15} = \underline{\underline{\frac{76}{15}}}$$

d)



$f(0) = 0$

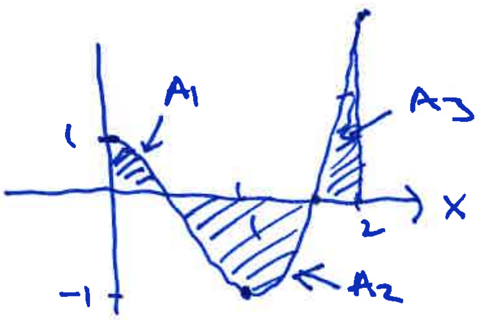
$\lim_{x \to 1} f(x) = -\infty$

$f(x) = 0.6\ln(1+x) + 0.4\ln(1-x)$

Problem Set 19:

4. $\int_0^2 x^3 - 3x + 1 \, dx = \left[\frac{1}{4}x^4 - 3\cdot\frac{1}{2}x^2 + x \right]_0^2$

$= \left(\frac{1}{4}\cdot 2^4 - \frac{3}{2}\cdot 2^2 + 2\right) - 0 = 4 - 6 + 2 = \underline{\underline{0}}$

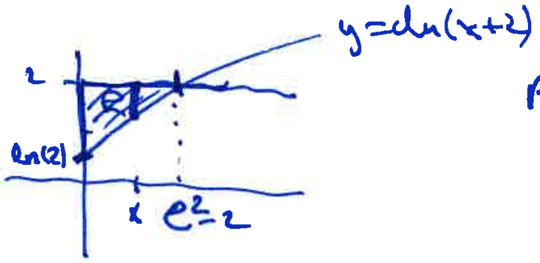


$\int_0^2 x^3 - 3x + 1 \, dx = A_1 - A_2 + A_3 = 0$

$A_2 = A_1 + A_3$

$f(x) = x^3 - 3x + 1$

5.



$A = \int_0^{e^2-2} 2 - \ln(x+2)\, dx = \int_2^{e^2} 2 - \ln(u)\, du$

$\boxed{\begin{aligned} u &= x+2 \\ du &= 1\cdot dx \end{aligned}}$

$x = 0 \to u = 2$

$x = e^2 - 2 \to u = e^2$

$= \left[2u - (u\ln u - u)\right]_2^{e^2} = \left[3u - u\ln u\right]_2^{e^2}$

$= (3e^2 - e^2\cdot 2) - (6 - 2\ln 2) = \underline{\underline{e^2 - 6 + 2\ln 2}}$

$f(x) = \ln(x+2)$

$\ln(x+2) = 2$

$x + 2 = e^2$

$x = \underline{e^2 - 2}$

9.

a) $\int 30x\sqrt{x}\,dx = 30\int x^{3/2}\,dx = 30\cdot\frac{2}{5}\cdot x^{5/2} + C$

$= \underline{12x^2\sqrt{x} + C}$

b) $\int xe^{-x}\,dx = \quad -e^{-x}\cdot x - \int(-e^{-x})\cdot 1\,dx$

$$\boxed{\begin{array}{ll} u = -e^{-x} & v = x \\ u' = e^{-x} & v' = 1 \end{array}}$$

$= -xe^{-x} + \int e^{-x}\,dx = \underline{-xe^{-x} - e^{-x} + C}$

c) $\int \frac{6-3x}{4-9x^2}\,dx = \int \frac{2}{\underset{\substack{\uparrow \\ u=2+3x}}{2+3x}} + \frac{1}{\underset{\substack{\uparrow \\ u=2-3x}}{2-3x}}\,dx$ Substitution

$4-9x^2 = (2+3x)(2-3x)$

~~$= -9(x-2/3)(x+2/3)$~~

$\frac{6-3x}{4-9x^2} = \frac{A}{2+3x} + \frac{B}{2-3x} \quad | \cdot CD$

$6-3x = A\cdot(2-3x) + B(2+3x)$

$6-3x = (2A+2B) + (-3A+3B)x$

$2A+2B = 6 \qquad A+B=3 \qquad \underline{B=3-A}$

$-3A+3B = -3 \qquad -3A+3(3-A) = -3$

$-6A+9 = -3$

$\frac{-6A}{-6} = \frac{-12}{-6}$

$\underline{A=2} \quad \underline{B=1}$

$= \frac{2}{3}\ln|2+3x| + \frac{1}{-3}\ln|2-3x| + C$

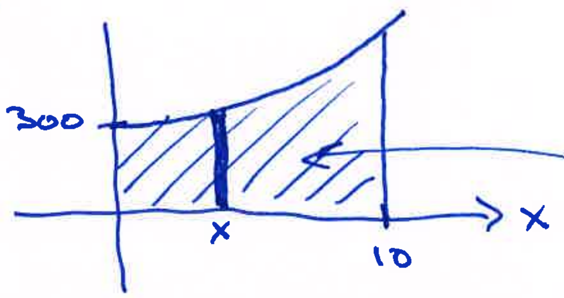
$= \underline{\frac{2}{3}\ln|2+3x| - \frac{1}{3}\ln|2-3x| + C}$

① Economic application of integration

Integration = sum things that change continuously

a) Cash flow:

Ex:



Cash flow:

300 MNOK / year

$f(x) = 300 \cdot 1{,}06^x$

Total income 10 first years = area under f(x) in [0,10]

$$= \int_0^{10} f(x)\,dx = \int_0^{10} 300 \cdot 1{,}06^x\,dx$$

$$= 300 \cdot \left[\frac{1}{\ln 1{,}06} \cdot 1.06^x \right]_0^{10}$$

$$= \frac{300}{\ln 1{,}06} \cdot (1.06^{10} - 1) \simeq \underline{\underline{4.072}} \text{ MNOK}$$

$$\int 1.06^x\,dx = \frac{1.06^x}{\ln 1.06} + C$$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad \text{when } a>0}$$

or $\displaystyle\int 1.06^x\,dx = \int e^{\ln(1.06^x)}\,dx$

$$= \int e^{x \cdot \ln 1{,}06}\,dx = \int e^u \cdot \frac{du}{\ln(1{,}06)} = \frac{1}{\ln(1{,}06)}\,e^u + C$$

$$\boxed{\begin{aligned} u &= \ln(1.06)x \\ du &= \ln(1.06)\cdot dx \end{aligned}}$$

$$= \frac{e^{\ln(1.06)x}}{\ln(1.06)} + C$$

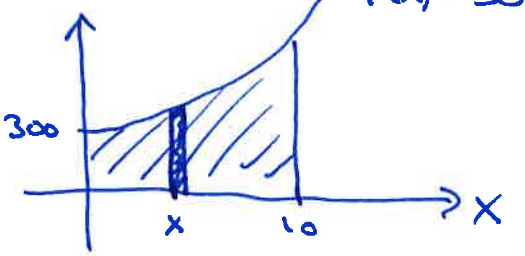
Cash flow: $f(x)$ MNOK/year

Total cash flow: $\int_0^T f(x)\,dx$

Present value: $\int_0^T f(x)\,e^{-rx}\,dx$

$f(x) = 300 \cdot 1.06^x$

e^{-rx}: continuous comp. with discount rate r



$f(x) \cdot dx$

$\frac{f(x)dx}{e^{rx}} \rightsquigarrow \int_0^{10} \frac{f(x)}{e^{rx}}\,dx$

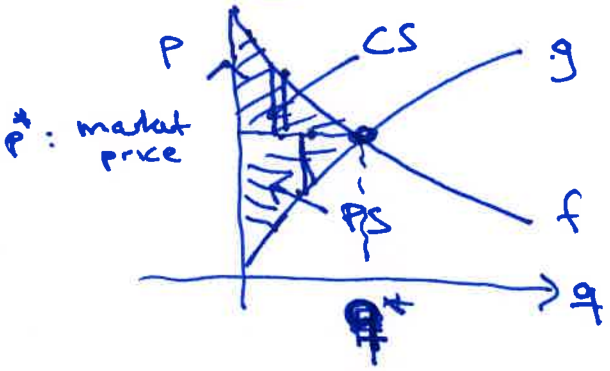
$$= \int_0^{10} \frac{300 \cdot 1.06^x}{e^{rx}}\,dx = 300 \int_0^{10} \frac{e^{x \cdot \ln(1.06)}}{e^{0.10x}}\,dx$$

$r = 0.10$

$$= 300 \int_0^{10} e^{(\ln 1.06 - 0.10)x}\,dx = \frac{300}{\ln 1.06 - 0.10} \Big[e^{(\ln 1.06 - 0.1)x} \Big]_0^{10}$$

$$= \frac{300}{\ln(1.06) - 0.10} \cdot \left(e^{10(\ln 1.06 - 0.10)} - 1 \right)$$

b) Consumer / producer surplus



p: price
q: quantity

$p = f(q)$ inverse demand fn.

$p = g(q)$ inverse supply fn.

CS: Consumer surplus

$$\int_0^{q^*} f(q) - p^* \, dq$$

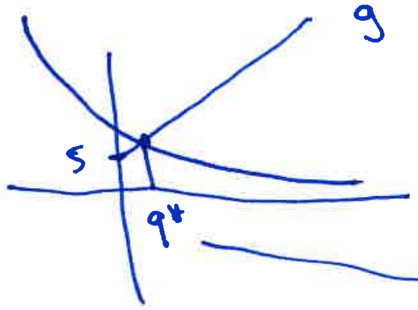
PS: Producer surplus

$$\int_0^{q^*} p^* - g(q) \, dq$$

Ex:

$$f(q) = \frac{3000}{q+50}$$

$$g(q) = q + 5$$



Market price:

$f(q) = g(q)$:

$$\frac{3000}{q+50} = q + 5$$

$$3000 = (q+5)(q+50)$$

$$3000 = q^2 + 55q + 250$$

$$0 = q^2 + 55q - 2750$$