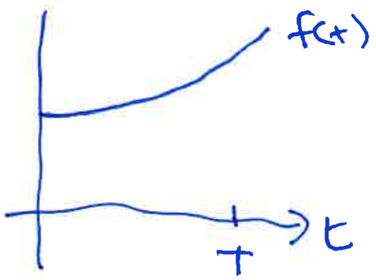
Plan
1 Linear systems of equations
2 Gaussian elimination

Review:

a) Continuous cash flows:

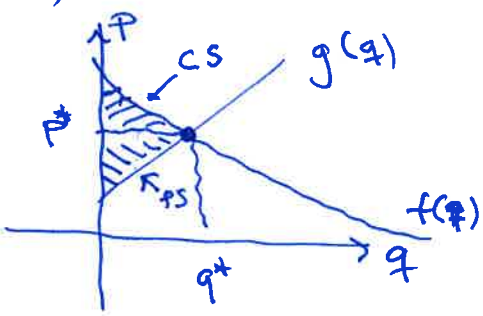


$f(t)$: cash flow at time t

$$\boxed{\begin{array}{ll}\text{Total cash flow:} & \int_0^T f(t)\,dt \\ \text{Total present value:} & \int_0^T f(t)\cdot e^{-rt}\,dt\end{array}}$$

(cont. comp., disc. rate r)

b)



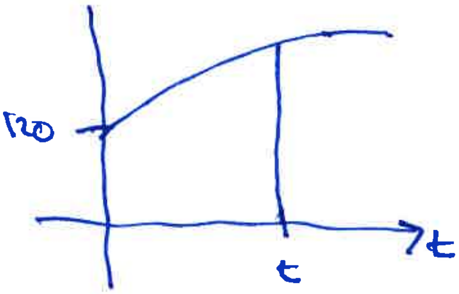
$p = f(q)$: inverse demand curve

$p = g(q)$: — " — supply "

$$\boxed{\begin{array}{l} CS = \int_0^{q^*} f(q) - p^*\,dq \\ PS = \int_0^{q^*} p^* - g(q)\,dq \end{array}}$$

Problem set 20, Pb. 6

$V(t) = 120\, e^{\sqrt{t}/5}$ value of a property at time t



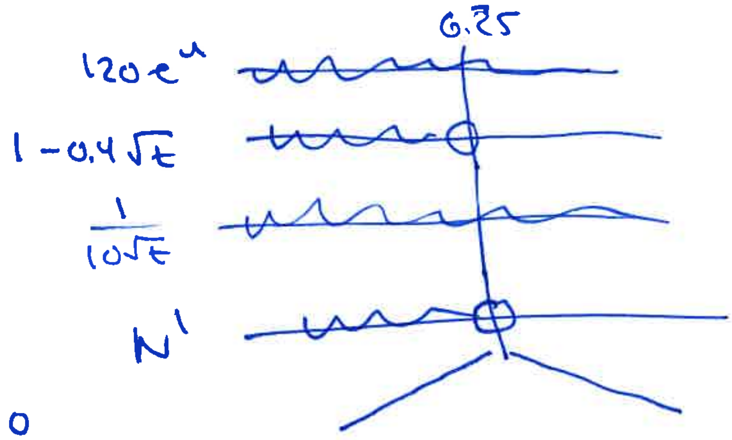
a) Maximize present value of sale price w/ disc. rate $r = 4\%$.

$$N(t) = \frac{V(t)}{e^{rt}} = \frac{120\, e^{\sqrt{t}/5}}{e^{0.04t}}$$
$$= 120\, e^{\sqrt{t}/5} \cdot e^{-0.04t}$$
$$= 120\, e^{u}, \quad u = \sqrt{t}/5 - 0.04t$$

$$N'(t) = 120\, e^{u} \cdot u'$$
$$= 120\, e^{u} \cdot \left(\frac{1}{5}\cdot\frac{1}{2\sqrt{t}} - 0.04\right)$$
$$= 120\, e^{u} \cdot \frac{1 - 0.04\cdot 10\sqrt{t}}{10\cdot\sqrt{t}}$$

$N' = 0$: $1 - 0.04 \cdot 10\sqrt{t} = 0$

$$0.4\sqrt{t} = 1$$
$$\sqrt{t} = 1/0.4 = 5/2$$
$$t = (5/2)^2 = 25/4$$
$$= \underline{6.25}$$



Present value is maximal after $\underline{\underline{t = 6.25}}$ years.

b) Doubling of the price:

$$V(t) = 120 \cdot e^{\sqrt{t}/5} = 240$$
$$e^{\sqrt{t}/5} = \frac{240}{120} = 2$$
$$\sqrt{t}/5 = \ln(2)$$
$$\sqrt{t} = 5\ln(2)$$
$$t = (5\ln 2)^2 = \underline{\underline{25(\ln 2)^2}}$$
$$T = 25(\ln 2)^2$$

$$120\, e^{\sqrt{t}/5} = 480$$
$$e^{\sqrt{t}/5} = \frac{480}{120} = 4$$
$$\sqrt{t}/5 = \ln 4$$
$$\sqrt{t} = 5\ln 4 = 5\cdot 2\ln 2$$
$$t = 25\cdot 4\cdot \ln(2)^2$$
$$= 4T$$

① Systems of equations

Ex: $\boxed{\begin{aligned} x+y&=8 \\ x-y&=4 \end{aligned}}$ $\quad y=8-x$, $\quad y=x-4$

$8-x=x-4$

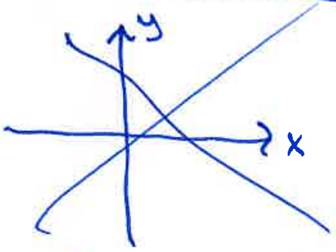
$8+4=2x$

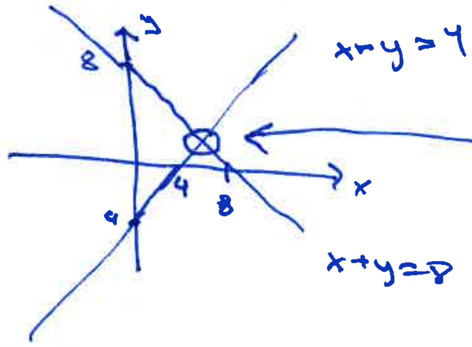
$2x=12$

$\underline{x=6}$

$\underline{y=2}$

Solution: $(x,y)=\underline{(6,2)}$





Solution to the system

A solution to a system of equations is a value for each variable that satisfies all eqn's simultaneously.

⇕

Intersection pts

Ex: $x^2+y^2=10$
$x+y=4$

Substitution:

$x+y=4 \Rightarrow y=\underline{4-x}$

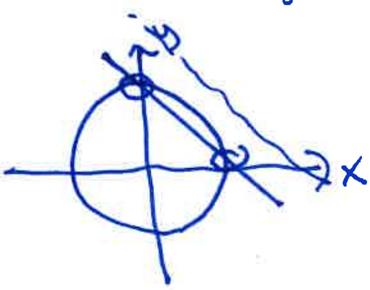
$x^2+y^2=10 \Rightarrow x^2+(4-x)^2=10$

$x^2+16-8x+x^2=10$

$2x^2-8x+6=0$

$(x-1)(x-3)=x^2-4x+3=0$

$x=1$, $y=3$, $x=3$, $y=1$



Solutions:

$(x,y)=\underline{(1,3),(3,1)}$

Linear systems of equations = linear systems

mxn linear system: m equations

n variables

all equations are linear

(first order)

Ex:

$x + y + z = 3$ (1)

$x + 2y + 4z = 7$ (2)

$x + 3y + 9z = 13$ (3)

3x3 linear system

Elimination:

(a) $x + y + z = 3$ (1)

(b) $y + 3z = 4$ (2)−(1)

(c) $2y + 8z = 10$ (3)−(1)

(a) $x + y + z = 3$

(b) $y + 3z = 4$

(c)−2(b) $2z = 2$

echelon form

$x = 1$

$y = 1$

$z = 1$

Backwards substitution

Substitution:

$x + y + z = 3 \Rightarrow x = 3 - y - z$ x=1

$(3-y-z) + 2y + 4z = 7$

$(3-y-z) + 3y + 9z = 13$

$\boxed{\begin{matrix} y + 3z = 4 \\ 2y + 8z = 10 \end{matrix}}$

$y + 3z = 4 \Rightarrow y = 4 - 3z$ y=1

$2(4-3z) + 8z = 10$

$\boxed{2z = 2}$

$z = 1$

Solution: $(x, y, z) = \underline{(1, 1, 1)}$

Defn: An $m \times n$ linear system in the variables $x_1, x_2, x_3, \ldots, x_n$ is a system of equations of the form:

$$m \begin{cases} a_{11} x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} \cdot x_n = b_m \end{cases}$$

where $a_{11}, \ldots, a_{mn}, b_1 \ldots, b_m$ are given numbers.

Defn: The coefficient matrix of the linear system is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and the extended / augmented matrix is

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Ex:

$$\begin{aligned} x+y-z &= 4 \\ x-y+2z &= 1 \\ x+2y+4z &= 28 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & 2 & 4 & 28 \end{array}\right)$$

3×3 linear system — augmented matrix

② Gaussian elimination:

A quick and systematic method for solving linear systems in general.

Ex:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & 2 & 4 & 28 \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & -3 \\ 1 & 2 & 4 & 28 \end{array}\right) \begin{pmatrix} (1) \\ (2)-(1) \\ (3) \end{pmatrix}$$

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Elementary row operation

Defn: Elementary row operations

i) To switch two rows

ii) Multiply a row with $c \neq 0$ $\quad \cdot c$

iii) Add a multiple of one row to another row $\quad c$

Any linear system can be solved using these operations

Defn: A pivot = first non-zero entry in a row

An echelon form is a matrix such that

i) All entries under a pivot are zero

ii) All zero rows are in the bottom of the matrix.

Ex:
$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & -3 \\ 1 & 2 & 4 & 28 \end{array}\right) \cdot (-1) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 1 & 5 & 24 \end{array}\right) \cdot 2$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 2 & 10 & 48 \end{array}\right) \cdot 1 \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 13 & 45 \end{array}\right)$$

echelon form

$$\begin{aligned} x + y - z &= 4 \\ -2y + 3z &= -3 \\ 13z &= 45 \end{aligned}$$

$\cancel{x = 4 - y + z = 4 + \frac{174}{13} - \frac{45}{13} = \dots}$

$-2y = -3 - 3(45/13) = \dfrac{-3 \cdot 13 - 3 \cdot 45}{13}$

$z = 45/13$

$\cancel{y = \frac{174}{13}}$

$y = \dfrac{174}{26} = \dfrac{87}{13}$

Solution: $\left(\dfrac{10}{13}, \dfrac{87}{13}, \dfrac{45}{13}\right)$

$x = 4 - \dfrac{174}{26} + \dfrac{45}{13}$

$= 20/26 = 10/13$

Gaussian elimination:

linear system $\longrightarrow$ augmented (extended) matrix

} elementary row operations

linear system $\longleftarrow$ echelon form

↓

backwards substitution

Ex:

$$x + y + z = 4$$
$$x - y - z = 0$$
$$x + 2y + 3z = 7$$

$\downarrow$

get zeros

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & 3 & 7 \end{array}\right) \begin{matrix} -1 \\ \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 1 & 2 & 3 \end{array}\right) \tfrac{1}{2}$$

get zero

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightarrow \begin{aligned} x + y + z &= 4 \\ -2y - 2z &= -4 \\ z &= 1 \end{aligned}$$

echelon form

$\Downarrow$

back substitution:

$z = 1$

$$\frac{-2y}{-2} = \frac{-4 + 2 \cdot 1}{-2} = \frac{-2}{-2}$$

$y = 1$

$x = 4 - 1 - 1 = 2$

Solution: $(x, y, z) = (2, 1, 1)$

Theorem:

Any $m \times n$ linear system has either

i) one solution (unique)

ii) no solution

iii) infinitely many solutions

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

echelon form

no solutions

(pivot in last column)

$$x+y+z=4$$
$$y+2z=6$$
$$0=4$$

impossible

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

z free variable

(z can be any number)

$$x+y+z=4$$
$$y+2z=6$$
$$\cancel{0=0}$$

$$y = 6-2z$$
$$x = 4-y-z$$
$$= 4-(6-2z)-z$$
$$x = -2+z$$

Solutions:

$(x,y,z) = (-2+z, 6-2z, z)$ with z free

infinitely many solutions