Plan

1. Linear systems and Gaussian elimination
2. Introduction to matrices and vectors
3. Determinants of quadratic matrices

1 Linear systems and Gaussian elimination

An $m \times n$ linear system:

$$m \text{ eqn's} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n} \cdot x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m}_{n \text{ variables}} \end{array} \right.$$

Gaussian elimination:

Method to solve linear systems

1. Write down the augmented matrix of the system.
2. Use elementary row operations until the matrix is in echelon form
3. Write down the linear system corresponding to the echelon form, and solve it by back substitution.

Ex:
$$\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ x+3y+9z&=13 \end{aligned}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} \\ -1 \\ -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ \\ -2 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\begin{aligned} x+y+z&=3 \\ y+3z&=4 \\ 2z&=2 \end{aligned}$$

$2z=2 \Rightarrow z=1$

$y+3z=4$

$y+3=4 \Rightarrow y=1$

$x+y+z=3$

$x+1+1=3 \Rightarrow x=1$

$(x,y,z)=(1,1,1)$

(one solution)

Some important facts about Gaussian elimination:

pivot position: the positions of pivots in an echelon form

Ex:
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

pivot positions: (1,1), (2,2), (3,3)

Facts:
- An echelon form is not unique, but the pivot positions are.
- the pivot positions determine the number of solutions of the linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

a) No solutions: (inconsistent) — there is a pivot in the last column

b) One solution: (consistent) — there is no pivot in the last column, and a pivot in all other columns (variable cols)

c) Infinitely many solutions: (consistent) — there is no pivot in the last column, and there is at least one variable column without pivot

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & -2 & -3 \\ 0 & 0 & 6 & 6 & 0 \end{array}\right)$$

x, y, z: basic
w: free

$x = -5w + 13$
$y = 5w - 3$
$z = -w$

free variable: no pivot pos.
basic variable: pivot pos.

no. degrees of freedom = number of free variable

Problem Sheet 21, Pb. 7

$$\begin{aligned} x + y + z + w &= 10 \\ x + 2y + 4z - w &= 7 \\ x - y + z + 11w &= 16 \end{aligned}$$

3×4 lin. sys.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & 2 & 4 & -1 & 7 \\ 1 & -1 & 1 & 11 & 16 \end{array}\right) \begin{array}{l}]-1 \\ \\ ]-1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & -2 & -3 \\ 0 & -2 & 0 & 10 & 6 \end{array}\right)]2$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & -2 & -3 \\ 0 & 0 & 6 & 6 & 0 \end{array}\right)$$

echelon form

$$\begin{aligned} \underline{x} + y + z + w &= 10 \\ \underline{y} + 3z - 2w &= -3 \\ 6\underline{z} + 6w &= 0 \end{aligned}$$

$$6z = -6w \iff \underline{z = -w}$$

$$y + 3z - 2w = -3$$

$$y + 3(-w) - 2w = -3$$

$$\Updownarrow$$

$$y = \underline{5w - 3}$$

$$x + y + z + w = 10$$

$$x + (5w - 3) + (\cancel{-w}) + \cancel{w} = 10$$

$$x = \underline{-5w + 13}$$

$$(x, y, z, w) = (-5w + 13,\ 5w - 3,\ -w,\ w)$$

where w is a free variable

(infinitely many solutions)

9. $2xy + y^3 + y^2 = 0 \quad (1)$

$x^2 + 3xy^2 + 2xy = 0 \quad (2)$

$(1) \quad y(2x + y^2 + y) = 0 \qquad y = 0 \quad \text{or} \quad 2x + y^2 + y = 0$

$(2) \quad x(x + 3y^2 + 2y) = 0 \qquad x = 0 \quad \text{or} \quad x + 3y^2 + 2y = 0$

(a) $y = 0, x = 0$: $\quad (x,y) = (0,0)$

(b) $y = 0, \; x + 3y^2 + 2y = 0$: $\quad (x,y) = (0,0)$

(c) $2x + y^2 + y = 0, \; x = 0$: $\quad (x,y) = (0,0), (0,-1)$

$x = 0, \; y^2 + y = 0$

$y(y+1) = 0 \Rightarrow y = 0, \; y = -1$

(d) $2x + y^2 + y = 0, \; x + 3y^2 + 2y = 0$:

$x = -3y^2 - 2y$

$2(-3y^2 - 2y) + y^2 + y = 0$

$-6y^2 - 4y + y^2 + y = 0$

$-5y^2 - 3y = 0$

$-y(5y + 3) = 0$

$y = 0 \quad \text{or} \quad y = -3/5$

$x = 0$

$x = -3(9/25) - 2(-3/5)$

$= \frac{-27}{25} + \frac{6 \cdot 5}{5 \cdot 5}$

$= \frac{30 - 27}{25} = 3/25$

Solutions: $(x,y) = (0,0), (0,-1), (3/25, -3/5)$

(2) Intro. to matrices and vectors

<u>Defn:</u> An <u>$m\times n$-matrix</u> is a rectangular array of numbers with <u>m rows</u>, n <u>cols</u>

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

2×3-matrix

a_{21}: row 2, col. 1

<u>Defn:</u> An <u>n-vector</u> is a matrix with n rows and 1 column.

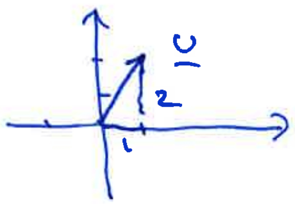
<u>Ex:</u> $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2-vector

$\underline{w} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 3-vector

$\underline{v} = \vec{v}$ = boldface v

<u>Geometric representation.</u>

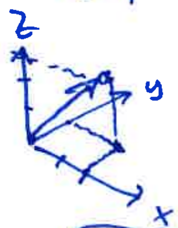
<u>n=2:</u> $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$$\|\underline{v}\| = \sqrt{1^2+2^2} = \sqrt{5} \approx \underline{2.23}$$

length of the vector $\underline{v}$

<u>n=3:</u> $\underline{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$



$$\|\underline{v}\| = \sqrt{2^2+1^2+3^2} = \underline{\underline{\sqrt{14}}}$$

③ Determinants:

$A \rightsquigarrow \det(A) = |A|$

$n \times n$-matrix (a number)

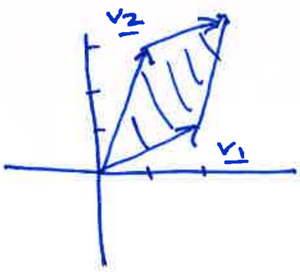
(square matrix)

(#rows = #cols)

n = 2:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\boxed{|A| = ad - bc}$

Ex: $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = A$ $|A| = 2 \cdot 3 - 1 \cdot 1 = \underline{\underline{5}}$



$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\|\underline{v_1}\| = \sqrt{2^2+1^2} = \sqrt{5}$

$\|\underline{v_2}\| = \sqrt{1^2+3^2} = \sqrt{10}$

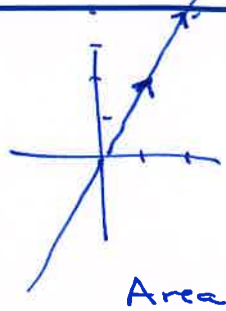
$|A| = \pm$ (area of parallelogram spanned by $\underline{v_1}$, $\underline{v_2}$)

$\|\underline{v_1}\| \cdot \|\underline{v_2}\| = \sqrt{5} \cdot \sqrt{10}$

$= \sqrt{50} \cong 7.1$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ $|A| = 1 \cdot 1 - 2 \cdot 3 = -5$

$|A|=0$: **Ex:** $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$



Area $=0$

$|A|=0 \Leftrightarrow \underline{v_1}, \underline{v_2}$ lie along the same line

$n\geq 3$: $A=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$

$$\boxed{|A| = \begin{matrix} aei+bfg+cdh \\ -ceg-bdi-afh \end{matrix}}$$

Cofactor expansion: (along a row or a column) method for computing determinants when $n\geq 3$

Ex:

$A=\begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix}$

Cofactor expansion along the first row

$$|A| = 1\cdot C_{11} + 1\cdot C_{12} + 1\cdot C_{13}$$
$$= 1\cdot(+1)\cdot M_{11} + 1\cdot(-1)\cdot M_{12} + 1\cdot(+1)\,M_{13}$$
$$= +1\cdot\begin{vmatrix}2&4\\3&9\end{vmatrix} - 1\cdot\begin{vmatrix}1&4\\1&9\end{vmatrix} + 1\cdot\begin{vmatrix}1&2\\1&3\end{vmatrix}$$
$$= 1\cdot(18-12) - 1\cdot(9-4) + 1\cdot(3-2)$$
$$= 6-5+1 = \underline{\underline{2}}$$

Cofactors:

$$C_{ij} = (-1)^{i+j}\cdot M_{ij}$$

M_{ij}: minor in pos (i,j) = determinant of the submatrix we get when we delete row i and col. j

3x3 case

Signs:

$$\begin{pmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \\ + & - & + & - & + & - & \cdots \\ \vdots & & & & & & \end{pmatrix}$$

Minors: $A=\begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix}$

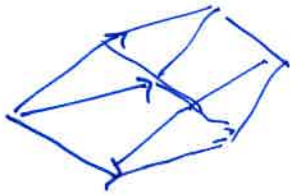
$$M_{11} = \begin{vmatrix}2&4\\3&9\end{vmatrix}$$

$$M_{12} = \begin{vmatrix}1&4\\1&9\end{vmatrix}$$

Facts:

- Cofactor expansion along any row or column gives the same result, $|A|$.
- If $n=3$:

$A \to$ three column vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$



parallelepiped

$|A| = \pm$ volume of the parallelepiped

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$|A| = -1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -(9-4) + 2(9-1) - 3(4-1)$$

$$= -5 + 16 - 3 \cdot 3 = \underline{\underline{2}}$$

Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 7 & 4 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 0 \cdot * - 0 \cdot *$$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 7 & 4 & 1 \end{pmatrix}$ $\quad |A| = \ldots$

$$= -(2-12) = \underline{\underline{10}}$$

Ex:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\left(+1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right) + 1 \cdot \left(-1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right)$$

$$= 1 \cdot 1 \cdot 0 + 1 \cdot (-1) \cdot 0 = \underline{\underline{0}}$$