Plan

1 Determinants and linear systems
2 Linear systems with parameters
3 Vector and matrix equations

Review:

a) Linear systems: $(A \mid \underline{b})$

Inconsistent (no solutions): pivot pos in the last column.

Consistent (there are solutions): free variables ↑ columns without pivot; basic (dependent) var. ↑ columns with pivots

no. degrees of freedom = no. free variables

0 degrees of freedom: one unique solution

at least one degree of freedom: infinitely many solutions

b) Determinants:

n=2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad |A| = ad - bc$

n≥3 (and in general): Cofactor expansion

along first row

Ex:

$$\begin{vmatrix} 2 & 0 & 7 \\ 4 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = +2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0 \cdot * + 7 \cdot \begin{vmatrix} 4 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 2(-1) + 7(4+1) = -2 + 35 = \underline{\underline{33}}$$

> * Cofactor expansion along any row or column gives the same result, $|A|$.
>
> * This method is general

Problem Sheet 22, 6a):

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 1 \cdot \left(+(-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right) + 1 \left(-1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right)$$

$$= 1 \cdot (-1) \cdot (-2) + 1 \cdot (-1) \cdot (-2) = \underline{\underline{4}}$$

① Determinants and linear systems

Eks: $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = E$

$|A| = |E| = 4$

$|E| = 1 \cdot 1 \cdot (-2) \cdot (-2)$
$= 4$

(product of the numbers on the diagonal)

$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \cdot 1 \cdot \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$

$= 1 \cdot 1 \cdot (-2) \cdot (-2) = 4$

Fact: The determinant of an echelon form is the product of the elements on the diagonal

$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & 0 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 1 \cdot 4 \cdot 0 \cdot 0 = 0$

Fact: Let $A \to B$ be an elementary row operation

1) If you switch two rows, then $|B| = -|A|$
2) If you multiply a row with $c \neq 0$, then $|B| = c \cdot |A|$
3) If you add a multiple of one row to another row, then $|B| = |A|$.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \cdot 3 \to C = \begin{pmatrix} 3 & 4 \\ 3 & 6 \end{pmatrix}]-1$

$|A| = 4 - 6 = -2$ $\quad |B| = 6 - 4 = 2$

$|C| = 18 - 12 = 6$

$\to E = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$

$|E| = 6$

$\cancel{\left|\begin{pmatrix} 3 & 4 \\ 3 & 6 \end{pmatrix}\right| = \left|\begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix}\right|}$

$\cancel{6 \quad -6}$

$\cancel{R(2) := R(1) - R(2)}$

Method: Computing determinants $|A|$ using Gaussian elimination

i) $A \to \dots \to E$ sequence of elementary row operations, to get an echelon form E

ii) If all elementary row operations are of Type 3, then $|A| = |E|$

iii) If there are elementary row operations of Type 1 and 2 as well, find $|A|$ using $|E|$.

Ex:

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 7 \\ 2 & 0 & 2 & 3 \\ 2 & 7 & 3 & 0 \end{vmatrix} \; 1 \; = \begin{vmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 7 \\ 2 & 0 & 2 & 3 \\ 2 & 7 & 3 & 0 \end{vmatrix} \; -1 \; -2 \; -2$$

$$= \begin{vmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -2 \\ 0 & -4 & -2 & -15 \\ 0 & 3 & -1 & -18 \end{vmatrix} \; -4 \; 3 \; = \begin{vmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -7 & -24 \end{vmatrix} \; 7/6$$

$$= 1 \cdot (-1) \cdot \begin{vmatrix} 6 & -7 \\ -7 & -24 \end{vmatrix}$$

$$= 1 \cdot (-1)\,(-144 \cancel{+} -49)$$

$$= \underline{\underline{193}}$$

$$= \begin{vmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & 0 & (-24 - 49/6) \end{vmatrix}$$

$$= 1 \cdot (-1) \cdot 6 \cdot \left(-24 - \frac{49}{6}\right)$$

$$= -6\left(-24 - \frac{49}{6}\right)$$

$$= 144 + 49 = \underline{\underline{193}}$$

Alt: Cofactor expansion

$$\begin{vmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 7 \\ 2 & 0 & 2 & 3 \\ 2 & 7 & 3 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & 7 \\ 2 & 2 & 3 \\ 2 & 3 & 0 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 & 7 \\ 2 & 0 & 3 \\ 2 & 7 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 2 & 7 & 3 \end{vmatrix}$$

$$= -1\big(1 \cdot (0-9) + 7 \cdot (6-4)\big) + 2\big(-2(0-49) - 3 \cdot (7-2)\big) - 2\big(1 \cdot (0-14) - 1(6-4)\big)$$

$$= -(-9+14) + 2(98-15) - 2(-14-2)$$

$$= -5 + 2 \cdot 83 + 32 = -5 + 166 + 32 = \underline{\underline{193}}$$

Linear systems:

Special case: $n \times n$ linear system (quadratic, # eqns = # var's)

$$n\left\{\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{array}\right.$$

$$A = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}}_{n}$$

$n \times n$ - matrix

coeff. matrix

Fact:

$|A| \neq 0$: the system has a unique solution

$|A| = 0$: — " — either no solutions or inf. many solutions

Echelon form:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x + 2y + 3z &= b_1 \\ 4y + 7z &= b_2 \\ 3z &= b_3 \end{aligned}$$

one solution

$$\begin{aligned} x + 2y + 3z &= b_1 \\ 4y + 7z &= b_2 \\ 0 &= b_3 \end{aligned}$$

$b_3 \neq 0$: no solutions

$b_3 = 0$: infinitely many solutions

2 Linear systems with parameters

Ex: $\boxed{\begin{aligned} x+y&=4 \\ x-ay&=2 \end{aligned}}$

x,y: variables
a: parameter

↓

solve for (x,y) for each value of a

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -a & 2 \end{array}\right) \rbrack -1 \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -a-1 & -2 \end{array}\right)$$

$a=-1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & -2 \end{array}\right)$$

no solutions

$a\neq -1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -a-1 & -2 \end{array}\right)$$

one solution

$$x+y=4$$
$$(-a-1)y=-2$$

$$y=\frac{-2}{-a-1}=\frac{2}{a+1}$$

$$x=4-\frac{2}{a+1}=\frac{4(a+1)-2}{a+1}$$
$$=\frac{4a+2}{a+1}$$

Solutions:
$$\begin{cases} (x,y)=\left(\frac{4a+2}{a+1}, \frac{2}{a+1}\right), & a\neq -1 \\ \text{no solution}, & a=-1 \end{cases}$$

Ex: $\begin{aligned} ax - y &= 4 \\ x - ay &= 2 \end{aligned}$ (a parameter)

$\left(\begin{array}{cc|c} a & -1 & 4 \\ 1 & -a & 2 \end{array}\right)$ possible, but cumbersome to do with Gaussian elimination

Alt: $A = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}$ $|A| = -a^2 + 1 = 1 - a^2$

$|A| = 0$: $1 - a^2 = 0$
$a = \pm 1$

a) $|A| \neq 0$: one solution $\rightarrow$ Kramer's rule
$(a \neq \pm 1)$

b) $|A| = 0$: no solutions or inf. many solutions
$(a = \pm 1)$

$a = 1$: $\left(\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & -1 & 2 \end{array}\right)$ $\rightarrow$ $\left(\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 0 & -2 \end{array}\right)$ no solutions

$a = -1$: $\left(\begin{array}{cc|c} -1 & -1 & 4 \\ 1 & 1 & 2 \end{array}\right)$ $\rightarrow$ $\left(\begin{array}{cc|c} -1 & -1 & 4 \\ 0 & 0 & 6 \end{array}\right)$ no solutions

Method: $n \times n$ linear systems with parameters

i) Use $|A|$ to separate { a) $|A| \neq 0$: one solution; b) $|A| = 0$: no or inf. many sol. }

ii) In case a), use Cramer's rule to find solution

iii) In case b), use Gaussian elimination to solve in each case

<u>Cramer's rule:</u> $\begin{cases} n\times n \text{ linear system} & (A \mid \underline{b}) \\ |A| \neq 0 & (\text{one solution}) \end{cases}$

$$x_1 = \frac{|A_1(\underline{b})|}{|A|}, \quad x_2 = \frac{|A_2(\underline{b})|}{|A|}, \quad \ldots\ldots, \quad x_n = \frac{|A_n(\underline{b})|}{|A|}$$

where $A_i(\underline{b})$ is the matrix we obtain by replacing col. i in A with $\underline{b}$.

<u>Ex:</u> $\begin{aligned} ax - y &= 4 \\ x - ay &= 2 \end{aligned}$

$$|A| = \begin{vmatrix} a & -1 \\ 1 & -a \end{vmatrix} = 1 - a^2$$

$$a \neq \pm 1: \quad |A| = 1 - a^2 \neq 0$$

$$x = \frac{|A_x(\underline{b})|}{|A|} = \underline{\underline{\frac{2-4a}{1-a^2}}}$$

$$A_x(\underline{b}) = \begin{pmatrix} 4 & -1 \\ 2 & -a \end{pmatrix}$$

$$|A_x(\underline{b})| = \begin{vmatrix} 4 & -1 \\ 2 & -a \end{vmatrix} = -4a + 2$$

$$y = \frac{|A_y(\underline{b})|}{|A|} = \underline{\underline{\frac{2a-4}{1-a^2}}}$$

$$|A_y(\underline{b})| = \begin{vmatrix} a & 4 \\ 1 & 2 \end{vmatrix} = 2a - 4$$

$\Downarrow$

$$\underline{\underline{(x,y) = \left(\frac{2-4a}{1-a^2}, \frac{2a-4}{1-a^2}\right)}}, \quad a \neq \pm 1$$

<u>Conclusion:</u> $\begin{cases} (x,y) = \left(\frac{2-4a}{1-a^2}, \frac{2a-4}{1-a^2}\right), & a \neq \pm 1 \\ \text{no solutions}, & a = \pm 1 \end{cases}$

Ex: $x+y+z=3$
$x+ay+4z=7$
$x+3y+az=a$

(a parameter)

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 4 \\ 1 & 3 & a \end{vmatrix} = 1\cdot(a^2-12) - 1\cdot(a-4) + 1\cdot(3-a)$$

$$= a^2-12-a+4+3-a$$

$$= \underline{a^2-2a-5}$$

$\underline{|A|=0}$: $a^2-2a-5=0$

$$a = \frac{2 \pm \sqrt{4+20}}{2}$$

$$a = \underline{1 \pm \sqrt{6}}$$

$\underline{a \neq 1 \pm \sqrt{6}}$: One solution

$$|A_1(\underline{b})| = \begin{vmatrix} 3 & 1 & 1 \\ 7 & a & 4 \\ a & 3 & a \end{vmatrix}$$

$$= 3\cdot(a^2-12) - 1\,(7a-4a) + 1\cdot(21-a^2)$$

$$= 3a^2-36-3a+21-a^2$$

$$= 2a^2-3a-15$$

$$x = \frac{2a^2-3a-15}{a^2-2a-5}$$

$$|A_2(\underline{b})| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 7 & 4 \\ 1 & a & a \end{vmatrix} = 1\cdot(7a-4a) - 3\cdot(a-4) + 1\cdot(a-7) = \underline{a+5}$$

$$y = \frac{a+5}{a^2-2a-5}$$

$$|A_3(\underline{b})| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & a & 7 \\ 1 & 3 & a \end{vmatrix} = 1\cdot(a^2-21) - 1\cdot(a-7) + 3\cdot(3-a) = a^2-4a-5$$

$$= 5a-5$$

$$z = \frac{a^2-4a-5}{a^2-2a-5}$$

3 Vector equations:

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\2\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\3\\0\end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix}4\\-1\\2\end{pmatrix}$

$$x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + x_3 \cdot \underline{v}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} = \underline{w}$$

vector equation: (x_1, x_2, x_3 variables)

$$x_1 \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\3\\0\end{pmatrix} + x_3 \cdot \begin{pmatrix}4\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

scalar mult. →

$$\begin{pmatrix}x_1\\x_1\\2x_1\end{pmatrix} + \begin{pmatrix}x_2\\3x_2\\0\end{pmatrix} + \begin{pmatrix}4x_3\\-x_3\\2x_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

add. →

$$\begin{pmatrix}x_1 + x_2 + 4x_3\\x_1 + 3x_2 - x_3\\2x_1 + 2x_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 = \underline{w}$

$$\left.\begin{aligned} x_1 + x_2 + 4x_3 &= 1\\ x_1 + 3x_2 - x_3 &= 0\\ 2x_1 \qquad + 2x_3 &= 0\end{aligned}\right\}\ 3\times 3 \text{ linear system}$$

$$\left(\underline{v}_1\ \underline{v}_2\ \underline{v}_3 \mid \underline{w}\right)$$

Why?

$$\underbrace{x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3}_{} = \underline{w}$$

linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$ = all expr. of this form for all possible choices of x_1, x_2, x_3

→ inconsistent (no sol.)
⇒ $\underline{w}$ is not a lin. comb. of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

→ consistent (at least one sol.)
⇒ $\underline{w}$ is a linear comb. of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

Geometry:

$2\underline{v}_1 + (-1)\underline{v}_2$: lin comb. of $\underline{v}_1, \underline{v}_2$