## Plan

1. Vector and matrix equations
2. Matrix algebra
3. Inverse matrices

Review:

i) Determinants using Gaussian elimination

If $A \to \dots \to E$ is a Gaussian process, where each elementary row operation is of "Type 3" (add a multiple of one row to another row), then $|A| = |E|$.

If $E$ is an echelon form, then $|E|$ is the product of the diagonal entries.

ii) Linear systems: $n \times n$ (quadratic) linear system (# eqn's = # var's) $\rightsquigarrow (A \mid \underline{b})$

$|A| \neq 0$: one solution
$|A| = 0$: no solutions or inf. many solutions

iii) Linear systems with parameters $\Big\{$ $n \times n$ lin. system with parameters

Method: 1) $|A|$ determines
a) $|A| \neq 0$: one solution
b) $|A| = 0$: no solutions or inf. many solutions

2) In case a): Cramer's rule

3) In case b): Use Gauss in each case (for each value of the parameters s.t. $|A| = 0$)

**Cramer's rule:** $\begin{cases} n\times n \text{ lin. system} \\ |A| \neq 0 \end{cases}$ $(A \mid \underline{b})$

$$x_1 = \frac{|A_1(\underline{b})|}{|A|}, \quad x_2 = \frac{|A_2(\underline{b})|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n(\underline{b})|}{|A|}$$

where $A_i(\underline{b})$ is the matrix you get when you replace column $i$ in $A$ with $\underline{b}$.

iv) **Vector equations:** $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r$ : $n$-vectors

$\underline{w}$ : $n$-vector

**Linear combination** of $\underline{v}_1, \dots, \underline{v}_r$: All expressions of the form

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_r \cdot \underline{v}_r$$

where $c_1, c_2, \dots, c_r$ are numbers.

$$\boxed{\underline{w} \text{ is a linear combination of } \underline{v}_1, \dots, \underline{v}_r} \iff x_1 \underline{v}_1 + x_2 \underline{v}_2 + \dots + x_r \cdot \underline{v}_r = \underline{w}$$

vector equation

$$\Updownarrow$$

Linear system with augmented matrix

$$\left( \underline{v}_1 \; \underline{v}_2 \; \cdots \; \underline{v}_r \;\middle|\; \underline{w} \right)$$

**Ex:**

vector eqn. $\rightarrow$ $x \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x + 3y \\ 2x - y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

lin sys. $\rightarrow$ $\begin{cases} x + 3y = 4 \\ 2x - y = 2 \end{cases}$

$$\left( \begin{array}{cc|c} 1 & 3 & 4 \\ 2 & -1 & 2 \\ \uparrow & \uparrow & \uparrow \\ \underline{v}_1 & \underline{v}_2 & \underline{w} \end{array} \right)$$

## Problem Sheet 23:

4b

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a\cdot(a^2-1) - 1\cdot(a-1) + 1\cdot(1-a)$$

$$= \underline{a(a^2-1)} - 2a + 2$$

$$= a\cdot(a-1)(a+1) - 2(a-1)$$

$$= (a-1)(a(a+1)-2) = (a-1)(a^2+a-2)$$

$$= (a-1)(a+2)(a-1) = \underline{(a-1)^2\cdot(a+2)}$$

b) $\underline{|A|=0}$: $a=1$, $a=-2$

$\underline{a=1}$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & -4 \end{array}\right)$ $\underline{a=-2}$: $\left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & -2 & -3 \end{array}\right)$

no solution.

a) $\underline{|A|\neq 0}$: $a \neq 1, -2$ → One solution, Cramer's rule

$$x = \frac{|A_1(b)|}{|A|} = \frac{a^2+a-2}{(a-1)^2(a+2)} = \underline{\underline{\frac{1}{a-1}}}$$

$$|A_1(\underline{b})| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & a & 1 \\ -3 & 1 & a \end{vmatrix} = 1\cdot(a^2-1) - 1\cdot(2a+3) + 1\cdot(2+3a)$$

$$= a^2+a-2$$

$$y = \frac{|A_2(\underline{b})|}{|A|} = \frac{2\cancel{(a-1)}\cancel{(a+2)}}{(a-1)^{\cancel{2}}\cancel{(a+2)}} = \underline{\underline{\frac{2}{a-1}}}$$

$$|A_2(\underline{b})| = \begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -3 & a \end{vmatrix} = a\cdot(2a+3) - 1\cdot(a-1) + 1\cdot(-5) = 2a^2+2a-4$$

$$= 2(a^2+a-2) = 2(a-1)(a+2)$$

$$z = \frac{|A_3(\underline{b})|}{|A|} = \frac{-3\cancel{(a-1)}\cancel{(a+2)}}{(a-1)^{\cancel{2}}\cdot\cancel{(a+2)}} = \underline{\underline{\frac{-3}{a-1}}}$$

$$|A_3(\underline{b})| = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 2 \\ 1 & 1 & -3 \end{vmatrix} = a(-3a-2) - 1\cdot(-5) + 1\cdot(1-a) = -3a^2-3a+6$$

$$= -3(a^2+a-2) = -3(a-1)(a+2)$$

8.

$$x_1 \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\2\\4\\3\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\-1\\1\\7\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

for which values of a, b, c, d does this eqn. have solutions

$$\Uparrow$$

$$\left(\begin{array}{ccc|c} \text{\textcircled{1}} & 1 & 1 & a \\ 1 & 2 & -1 & b \\ 1 & 4 & 1 & c \\ 1 & 3 & 7 & d \end{array}\right) \begin{array}{l} -1 \\ -1 \\ -1 \end{array} \to \left(\begin{array}{ccc|c} \text{\textcircled{1}} & 1 & 1 & a \\ 0 & \text{\textcircled{1}} & -2 & b-a \\ 0 & 3 & 0 & c-a \\ 0 & 2 & 6 & d-a \end{array}\right) \begin{array}{l} -3 \\ -2 \end{array}$$

$$\to \left(\begin{array}{ccc|c} \text{\textcircled{1}} & 1 & 1 & a \\ 0 & \text{\textcircled{1}} & -2 & b-a \\ 0 & 0 & \text{\textcircled{6}} & c-a-3(b-a) \\ 0 & 0 & 10 & d-a-2(b-a) \end{array}\right) \begin{array}{l} \leftarrow 2a-3b+c \\ \leftarrow a-2b+d \end{array} \quad -\tfrac{10}{6}$$

$$\left(\begin{array}{ccc|c} \text{\textcircled{1}} & 1 & 1 & a \\ 0 & \text{\textcircled{1}} & -2 & b-a \\ 0 & 0 & \text{\textcircled{6}} & 2a-3b+c \\ 0 & 0 & 0 & \underset{*}{\underset{\shortparallel}{a-2b+d-\tfrac{10}{6}(2a-3b+c)}} \end{array}\right)$$

$* = 0$: one solution
$* \neq 0$: no solutions $\Bigg\}$ $\underline{b}$ is a lin. comb. of $\underline{u}_1, \underline{u}_2, \underline{u}_3$

$$\Uparrow$$

$$* = 0: \quad a-2b+d = \frac{10}{6}(2a-3b+c)$$

$$6a-12b+6d = 10(2a-3b+c)$$

$$\underline{-14a+18b-10c+6d=0}$$

$(a,b,c,d) = (0,0,1,1)$: not a linear comb. since $-10+6 \neq 0$

**9.**

| | A | B | C |
|---|---|---|---|
| | 60 | 75 | 320 |
| 1 | 80 | 80 | 350 |
| 2 | 100 | 25 | 500 |
| 3 | 40 | 100 | 55 |

| | A | B | C |
|---|---|---|---|
| 1 | 20 | 5 | 30 |
| 2 | 40 | -50 | 180 |
| 3 | -20 | 25 | -265 |

$$\begin{pmatrix} 80 & 80 & 350 \\ 100 & 25 & 500 \\ 40 & 100 & 55 \end{pmatrix} - \begin{pmatrix} 60 & 75 & 320 \\ 60 & 75 & 320 \\ 60 & 75 & 320 \end{pmatrix} =$$

$C = 400.000$:

$$20x + 5y + 30z = R_1$$
$$40x - 50y + 180z = R_2$$
$$-20x + 25y - 265z = R_3$$
$$60x + 75y + 320z = C$$

b) $R_1 = 50'$, $R_2 = 25'$, $R_3 = -100'$, $C = 400'$

$$\begin{array}{c} \\ -2 \\ +1 \\ -3 \end{array} \left(\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right) \to \dots \to \left(\begin{array}{ccc|c} \boxed{20} & 5 & 30 & R_1 \\ 0 & \boxed{-60} & 120 & R_2 - 2R_1 \\ 0 & 0 & \boxed{-175} & R_3 + \frac{1}{2}R_2 \\ 0 & 0 & 0 & \boxed{*} \end{array}\right)$$

$$* = \underline{C - 5R_1 + 2R_2 + 2R_3}$$

$* \neq 0$: no solution

$* = 0$: One solution $(x,y,z)$ $\Leftrightarrow$ $\boxed{5R_1 - 2R_2 - 2R_3 = 400'}$

$R_1 = 50'$, $R_2 = 25'$, $R_3 = -100'$ : $\boxed{\text{ok}}$

c) $R_1 > 0$, $R_2 = R_3 = 0$ : $5R_1 - 2 \cdot 0 - 2 \cdot 0 = 400.000$

$\underline{R_1 = 80.000,\ R_2 = R_3 = 0}$

d) For ex: $R_1 = R_2 = R_3$

$5R_1 - 2R_1 - 2R_1 = 400' \Rightarrow R_1 = R_2 = R_3 = \underline{400.000}$

# ① Matrices and matrix algebra

## i) Addition / subtraction:

$A+B$, $A-B$ is defined if $A$ and $B$ have the same size

Ex:

$$\begin{pmatrix}1&2\\3&-1\end{pmatrix}+\begin{pmatrix}2&0\\7&-1\end{pmatrix}=\begin{pmatrix}1+2&2+0\\3+7&-1+(-1)\end{pmatrix}=\underline{\begin{pmatrix}3&2\\10&-2\end{pmatrix}}$$

$$\begin{pmatrix}1&2\\3&-1\end{pmatrix}-\begin{pmatrix}2&0\\7&-1\end{pmatrix}=\underline{\begin{pmatrix}-1&2\\-4&0\end{pmatrix}}$$

(vectors: special case)

## ii) Scalar multiplication:

$r\cdot A = A\cdot r$ ($r$ = a number)

Ex: $$3\cdot\begin{pmatrix}1&2\\3&-1\end{pmatrix}=\begin{pmatrix}3\cdot1&3\cdot2\\3\cdot3&3\cdot(-1)\end{pmatrix}=\underline{\underline{\begin{pmatrix}3&6\\9&-3\end{pmatrix}}}$$

(vectors: special case)

## iii) Matrix multiplication:

$$\underset{m\times n}{A}\cdot\underset{n\times p}{B}\rightsquigarrow\underset{m\times p}{AB}$$

* $A\cdot B$ is defined if #cols of $A$ = # rows in $B$

* the result has dimension #rows of $A$ × #cols in $B$

Ex: $$\begin{pmatrix}1&2\\3&-1\end{pmatrix}\cdot\begin{pmatrix}7\\0\end{pmatrix}=\begin{pmatrix}1\cdot7+2\cdot0\\3\cdot7+(-1)\cdot0\end{pmatrix}=\underline{\underline{\begin{pmatrix}7\\21\end{pmatrix}}}$$

$$\textcircled{2}\times\textcircled{2}=\textcircled{2}\times\textcircled{1}$$

$2\times1$

Ex: $$\underset{2\times 2}{\begin{pmatrix}2&0\\1&3\end{pmatrix}}\cdot\underset{2\times 2}{\begin{pmatrix}4&1\\-1&7\end{pmatrix}} = \underline{\underline{\begin{pmatrix}8&2\\1&22\end{pmatrix}}}$$

$$\begin{array}{c|c} & \begin{pmatrix}4&1\\-1&7\end{pmatrix}\\ \hline \begin{pmatrix}2&0\\1&3\end{pmatrix} & \begin{pmatrix}8&2\\1&22\end{pmatrix}\end{array}$$

Note: $\boxed{A\cdot B \neq B\cdot A}$

## Linear systems:

Ex:
$$\begin{aligned} x+y+z+w &= 3\\ 2x-y+3z &= 7\\ x+2y+4z-w &= 10\end{aligned}$$

$$(A|\underline{b}) = \left(\begin{array}{cccc|c} 1&1&1&1&3\\ 2&-1&3&0&7\\ 1&2&4&-1&10\end{array}\right)$$
with $\underbrace{\phantom{xxxx}}_{A}$ under the first four columns and $\underbrace{\phantom{x}}_{\underline{b}}$ under the last.

$$A=\begin{pmatrix}1&1&1&1\\2&-1&3&0\\1&2&4&-1\end{pmatrix} \qquad \underline{x}=\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \qquad \underline{b}=\begin{pmatrix}3\\7\\10\end{pmatrix}$$

$$A\cdot\underline{x} = \underset{3\times 4}{\begin{pmatrix}1&1&1&1\\2&-1&3&0\\1&2&4&-1\end{pmatrix}}\cdot\underset{4\times 1}{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}} = \begin{pmatrix}x+y+z+w\\2x-y+3z\\x+2y+4z-w\end{pmatrix}$$

$$= \begin{pmatrix}3\\7\\10\end{pmatrix} = \underline{b}$$

Matrix form of the linear system:

$$A\cdot\underline{x} = \underline{b}$$

## ③ Inverse matrices

**Identity matrix:** $I$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}$$

**Property:** $A \cdot I = A$, $IA = A$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 7 & 3 & 1 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 3 & 1 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

**Defn:** Let $A$ be an $n \times n$-matrix. The inverse matrix of $A$ is a matrix $A^{-1}$ such that

$$A^{-1} \cdot A = I \quad \text{and} \quad A \cdot A^{-1} = I$$

**Ex:** $A = \begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & +7 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Ex:**

$$2x = 6$$
$$\frac{1}{2}(2x) = \frac{1}{2} \cdot 6$$
$$\left(\frac{1}{2} \cdot 2\right) \cdot x = 3$$
$$1 \cdot x = 3$$
$$x = 3$$

**Want:**

$$A \cdot \underline{x} = \underline{b}$$
$$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$$
$$(A^{-1} \cdot A)\underline{x} = A^{-1}\underline{b}$$
$$I \cdot \underline{x} = A^{-1} \cdot \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$

**General results:** $A$ $n\times n$-matrix

1) $A^{-1}$ exists $\iff$ $|A| \neq 0$
2) If $|A|\neq 0$, there is a unique inverse matrix $A^{-1}$.

**The case $n=2$:**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = ad-bc$$

**$ad-bc=0$** : $A^{-1}$ does not exist

**$ad-bc\neq 0$:**

$$A^{-1} = \frac{1}{ad-bc}\cdot \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\text{adj}(A)}$$

(adjungated matrix)

**Ex:** $A = \begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}$

$$|A| = 1\cdot(-1) - 7\cdot 0 = -1$$

$$A^{-1} = \frac{1}{-1}\cdot\begin{pmatrix} -1 & -7 \\ 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 7 \\ 0 & -1 \end{pmatrix}}}$$

**The general case:** $n\geq 3$

**Ex:** $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

**$|A|\neq 0$:** $A^{-1} = \frac{1}{|A|}\cdot \text{adj}(A)$

**$|A|=0$** : $A^{-1}$ does not exist

**In general:**

$$\underset{n\times n\text{-matrix}}{A} \rightsquigarrow \underset{\text{cofactor matrix}}{C} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

$$A = \begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ a_{n1} \quad a_{n2} \quad \cdots \quad a_{nn} \end{pmatrix}$$

$$|A| = a_{11}\cdot C_{11} + a_{12}\cdot C_{12} + \cdots + a_{1n}\cdot C_{1n}$$

**Ex:**

$$A = \begin{pmatrix} \boxed{1 \quad 1 \quad 1} \\ 1 \quad 2 \quad 4 \\ 1 \quad 3 \quad 9 \end{pmatrix} \qquad |A| = \underline{1}\cdot 6 + \underline{1}\cdot(-5) + \underline{1}\cdot 1 = \underline{\underline{2}}$$

$$\left.\begin{matrix} C_{11} = +6 & C_{12} = -5 & C_{13} = +1 \\ C_{21} = -6 & C_{22} = +8 & C_{23} = -2 \\ C_{31} = +2 & C_{32} = -3 & C_{33} = +1 \end{matrix}\right\} \quad \underline{\underline{C = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}}}$$

$$\operatorname{adj}(A) = C^T = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\cdot \operatorname{adj}(A) = \frac{1}{2}\cdot\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}}}$$

**Transpose:**

$$\underset{m\times n\text{ matrix}}{A} \rightsquigarrow \underset{\substack{\text{transpose of } A,\\ n\times m\text{-matrix}}}{A^T}$$

rows of $A^T$ = cols of $A$

cols of $A^T$ = rows of $A$

||

reflexion along the diagonal