Plan

1. Matrix algebra
2. Exam problem 05/2017

Exam problems now available on the web page (link to exams)
- one translated to english
- many more in norwegian

Review:

a) Matrix multiplication:

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB}$$

Noncommutative: $AB \neq BA$

Ex:

$$\underset{3\times 3}{\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \cdot \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -1 \end{pmatrix}} = \begin{pmatrix} 13 & -3 \\ 2 & 15 \\ 0 & 8 \end{pmatrix}$$

b) Transpose:

$$\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$$

Ex: $\begin{pmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -1 \end{pmatrix}^T$ 3×2

$= \begin{pmatrix} 1 & 3 & 4 \\ 0 & 7 & -1 \end{pmatrix}$ 2×3

Defn: A matrix A is called symmetric if $A^T = A$.

Ex: $A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 7 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & -1 \\ 4 & -1 & 7 \end{pmatrix}$

A is symmetric

Eivind Eriksen, Office B4-032, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/EBA2910/

c) Inverse matrix:

A ($n \times n$)

<u>Defn:</u> An inverse matrix of A is a matrix A^{-1} such that

$$A \cdot A^{-1} = I \quad \text{and} \quad A^{-1} \cdot A = I$$

The matrix I is the identity matrix. It has the form

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and it has the property that

$$A \cdot I = A \quad \text{and} \quad I \cdot A = A$$

Results:

i) A^{-1} exists $\Longleftrightarrow$ $|A| \neq 0$
(A invertible)

ii) If A^{-1} exists, it is unique and it is given by

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}}_{\text{cofactor matrix of } A}{}^{T}$$

Application:

Let $A \cdot \underline{x} = \underline{b}$ be a linear system on matrix form.

<u>Assume $|A| \neq 0$</u> (and the system is $n \times n$):

$$A \cdot \underline{x} = \underline{b} \quad | A^{-1} \cdot$$
$$A^{-1} \cdot A \underline{x} = A^{-1} \cdot \underline{b}$$
$$I \cdot \underline{x} = A^{-1} \cdot \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$

<u>Remember:</u>

$|A| \neq 0$: one solution

$|A| = 0$: no / inf. many solutions

Use Gauss

Problem Sheet 24.

4. $A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$ $(A \cdot \underline{x} = \underline{b})$

a) $t=2$: $\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right) \rightarrow$ Gauss

b) $|A| = \begin{vmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = t(t^2-1) = t(t+1)(t-1)$

$|A|=0$: no sol. or inf. many s. $t=0$, $t=-1$, $t=1$

$|A| \neq 0$: one solution. $t \neq 0,-1,1$

$t=0$: $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ inf. many sol. $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$, y free

$t=1$: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ inf. many sol. $\begin{pmatrix} 1-z \\ 0 \\ z \end{pmatrix}$, z free

$t=-1$: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{1} \left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$ no sol.

Concl: $\begin{cases} \text{no sol.} & t=-1 \\ \text{inf. many s.} & t=0,1 \\ \text{one sol.} & t \neq 0,1,-1 \end{cases}$

c) $|A| = t(t+1)(t-1)$ $\qquad |A| \neq 0 \iff t \neq 0, 1, -1$

$t \neq 0, -1, 1$:

$$A^{-1} = \frac{1}{t(t+1)(t-1)} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T =$$

$$A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$$

$C_{11} = t^2 \quad C_{12} = 0 \quad C_{13} = -t$

$C_{21} = 0 \quad C_{22} = t^2 - 1 \quad C_{23} = 0$

$C_{31} = -t \quad C_{32} = 0 \quad C_{33} = t^2$

A symmetric

$\Downarrow$

C symmetric

$$A^{-1} = \frac{1}{t(t+1)(t-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$$

$A\underline{x} = \underline{b}$

$\cancel{A^{-1}A}\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{t(t+1)(t-1)} \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix} \cdot \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

$$= \frac{1}{t(t+1)(t-1)} \cdot \begin{pmatrix} t^3 - t^2 \\ 0 \\ -t^2 + t^3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{t^3 - t^2}{t(t+1)(t-1)} \\ 0 \\ \dfrac{t^{\cancel{2}}\cancel{(t-1)}}{\cancel{t}(t+1)\cancel{(t-1)}} \end{pmatrix} = \begin{pmatrix} t/(t+1) \\ 0 \\ t/(t+1) \end{pmatrix}$$

for $t \neq 0, -1, 1$

5 e) $(BAB^{-1})^2 \cdot B^2 = (BA\cancel{B^{-1}})(\cancel{B}AB^{-1}) \cdot \cancel{B} \cdot B$

$= BAAB = \underline{\underline{BA^2B}}$

① Matrix algebra.

Determinants:

i) $\boxed{|AB| = |A| \cdot |B|}$

ii) $|c \cdot A| = c^n \cdot |A|$ $\Big\}$ c number, A $n \times n$-matrix

iii) $|A^T| = |A|$

iv) $|A^{-1}| = \frac{1}{|A|}$

$A \cdot A^{-1} = I$

$\Downarrow$

$|A \cdot A^{-1}| = |I|$

$|A| \cdot |A^{-1}| = 1$

Transpose:

i) $(A \pm B)^T = A^T \pm B^T$

iv) $(A^T)^T = A$

ii) $(c \cdot A)^T = c \cdot A^T$

iii) $\boxed{(A \cdot B)^T = B^T \cdot A^T}$

Inverses:

i) $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

ii) $(A^{-1})^{-1} = A$

$A\cancel{B} \cdot \cancel{B^{-1}} A^{-1}$

$= \cancel{A} \cdot \cancel{A^{-1}} = I$

$\cancel{AB \cdot A^{-1} B^{-1}}$

5b) $(A^T \cdot A)^T = A^T \cdot (A^T)^T = \underline{\underline{A^T \cdot A}}$

(2) Exam MET11803, 05/2017, Question 1: $A \cdot \underline{x} = \underline{b}$

$$A = \begin{pmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3+a \\ a^2 \\ 3-a \end{pmatrix}$$

a) $a=1$:

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 0 & 2 & -3 \\ 0 & 2 & 2 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & -3 \end{array}\right) \qquad \begin{aligned} 2x + 2y &= 4 \\ 2y + 2z &= 2 \\ 2z &= -3 \end{aligned}$$

One solution:

$$\boxed{(x,y,z) = (-1/2, 5/2, -3/2)}$$

$z = \underline{-3/2}$

$2y = 2 - 2(-3/2) = 5 \Rightarrow y = \underline{5/2}$

$2x = 4 - 2(5/2) = -1 \Rightarrow x = \underline{-1/2}$

b) $a=1$:

$|A| = 2(0) - 2(4) = -8 \neq 0$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

A symm. $\Rightarrow$ C symm.

$$A^{-1} = \frac{1}{(-8)} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{(-8)} \cdot \begin{pmatrix} 0 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 0 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}}$$

$\underline{a=1}$: $A \cdot \underline{x} = \underline{b}$

$A^{-1} \cdot A \underline{x} = A^{-1} \underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b} = \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}}_{\underline{b}} = \frac{1}{2}\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1/2 \\ 5/2 \\ -3/2 \end{pmatrix}}}$$

c) $\boxed{A\underline{x} = \underline{b} \text{ one solution} \iff |A| \neq 0}$

$$|A| = \begin{vmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{vmatrix}$$

$= (1+a) \cdot ((1+a)^2 - 4) - 2 \cdot (2(1+a) - 2(1-a))$
$+ (1-a) \cdot (4 - (1+a)(1-a))$

$= (1+a)(a^2+2a-3) - 8a + (1-a)(a^2+3)$
$= a^2 + 2a - \cancel{3} + \cancel{a^3} + 2a^2 - 3a - 8a + a^2 + \cancel{3} - \cancel{a^3} - 3a$
$= \underline{4a^2 - 12a} = 4a(a-3)$

$|A| = 0$: $4a(a-3) = 0 \iff a = 0, a = 3$

$|A| \neq 0$: $a \neq 0, 3$

$\boxed{\text{Conclusion: } A\underline{x} = \underline{b} \text{ has one solution for } a \neq 0, 3}$

d) $A\underline{x}=b$ inconsistent (no solutions) $\Longrightarrow$ $|A|=0$

$\underline{a=0}$ or $\underline{a=3}$ (from c)

$\underline{a=0}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right) \begin{array}{l} \cdot(-2),\ \cdot(-1) \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$\underline{\text{inf. many sol.}}$

$\underline{(z \text{ free})}$

$\underline{a=3}$:

$$\left(\begin{array}{ccc|c} 4 & 2 & -2 & 6 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right) \cdot \tfrac{1}{2}$$

$$\to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right) \begin{array}{l} -1,\ +1 \end{array} \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 3 & 3 & 3 \end{array}\right) \cdot(-1)$$

$$\to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

no solutions

$\underline{\text{Concl:}}$ $\boxed{A\underline{x}=b \text{ inconsistent} \quad \underline{\underline{a=3}}}$

Exam	MET 11803 Matematikk for siviløkonomer
Date	May 30th 2017 at 0900 - 1400

This exam consists of 16+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 96p (16 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.
We consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3+a \\ a^2 \\ 3-a \end{pmatrix}$$

We consider a as a parameter and x,y,z as variables.

(a) **(6p)** Use Gaussian elimination to solve the linear system when $a = 1$.
(b) **(6p)** Find A^{-1} when $a = 1$, and use A^{-1} to solve the linear system in this case.
(c) **(6p)** Determine the values of a such that the linear system has exactly one solution.
(d) **(6p)** Determine the values of a such that the linear system is inconsistent.

Question 2.
Compute the integrals:

$$\text{(a) } \textbf{(6p)} \int_0^1 \frac{3}{(2-x)^4}\,dx \qquad \text{(b) } \textbf{(6p)} \int \frac{2e^x}{e^x + e^{-x}}\,dx \qquad \text{(c) } \textbf{(6p)} \int \frac{\ln x}{x^2}\,dx$$

Question 3.
We consider the function $f(x) = \dfrac{\ln x}{x^2}$ defined for $x > 0$.

(a) **(6p)** Find the maximum and minimum value of f, if they exist.
(b) **(6p)** Make a rough sketch of the area R in the first quadrant bounded by the graph of f, the x-axis, and the line $x = 1$, and compute the area of R.

Question 4.
We consider the function

$$f(x,y) = \frac{2x + 3y - 6}{xy}$$

(a) **(6p)** Compute f'_x og f'_y, and find the stationary points of f.
(b) **(6p)** Classify the stationary points as locale maxima, local minima or saddle points.
(c) **(6p)** Find the maximum and minimum value of f, if they exist.
(d) **(6p)** Show that the level curve $f(x,y) = 5$ intersects the line $y = 1$ in one point, and find the tangent to the level curve at this point.

Question 5.
We consider the Lagrange problem

$$\max / \min f(x,y) = x^4 + 2x^2y^2 - 4y^3 \quad \text{when} \quad x^2 + y^2 = 25$$

(a) **(6p)** Make a sketch of the curve given by $x^2 + y^2 = 25$, and determine if this is a bounded set.
(b) **(6p)** Write down the Lagrange conditions, and find all $(x,y;\lambda)$ that satisfy these conditions.
(c) **(6p)** Solve the Lagrange problem.

We change the constraint in the Lagrange problem to $x^2 + y^2 = 26$.

(d) **Extra credit (6p)** Estimate the maximum and minimum value of the new Lagrange problem.

Computation of determinants:

i) If A has a row (or column) of zeros, then $|A|=0$.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & -1 & \sqrt{2} \end{vmatrix} = 0$

ii) If A has two identical rows (or columns), then $|A|=0$.

Ex: $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} \\ \\ -1 \end{matrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

$1\cdot(7-12)-2\cdot(0)$
$+1\cdot(12-7) = -5$
$+5 = 0$

$\underset{A}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 1 & 4 & 1 \end{vmatrix}} = \underset{A^T}{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 3 & 1 \end{vmatrix}} \begin{matrix} \\ \\ \cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$

iii) If A has a row that is a linear comb. of the other rows, then $|A|=0$.
If A has a column that is a linear comb of the other cols, then $|A|=0$.

Ex: $\begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 2 & 2 & 6 \end{vmatrix} \begin{matrix} \\ \\ -2 \end{matrix} = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Ex:

$$\begin{array}{c} \begin{vmatrix} 1 & 3 & 4 \\ 7 & -1 & 6 \\ 4 & 2 & 6 \end{vmatrix} = 0 \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{array} \end{array}$$

$\underline{v}_3 = \underline{v}_1 + \underline{v}_2$

$R(3) = R(1) + R(2)$

$$\underbrace{\begin{vmatrix} 1 & 3 & 4 \\ 7 & -1 & 6 \\ 4 & 2 & 6 \end{vmatrix}}_{A} = \underbrace{\begin{vmatrix} 1 & 7 & 4 \\ 3 & -1 & 2 \\ 4 & 6 & 6 \end{vmatrix}}_{A^T} \begin{matrix} \\ -3 \\ -4 \end{matrix} = \begin{vmatrix} 1 & 7 & 4 \\ 0 & -22 & -10 \\ 0 & -22 & -10 \end{vmatrix} = 0$$

$S(1) = R(1)$

$S(2) = R(2) - 3R(1)$

$S(3) = R(3) - 4R(1)$

$S(2) = S(3)$

$R(2) - 3R(1) = R(3) - 4R(1)$

$R(2) - 3R(1) + 4R(1) = R(3)$

$\underline{R(1) + R(2) = R(3)}$

Alternatively:

$$\begin{vmatrix} 1 & 3 & 4 \\ 7 & -1 & 6 \\ 4 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 4 \\ 3 & -1 & 2 \\ 4 & 6 & 6 \end{vmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} = \begin{vmatrix} 1 & 7 & 4 \\ 3 & -1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

new row three

$= R(3) - R(1) - R(2) = \underline{0}$

Eivind Eriksen, Office B4-032, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/EBA2910/
BI Norwegian Business School
BSc in Business Analytics