Plan

1. Functions of two variables
2. Graphs and level curves
3. Linear functions
4. Partial derivation

EBA 29101 : Retake
EBA 29103 : Course paper.
– available from 13/03

Review:

a) $|AB| = |A| \cdot |B|$
$(AB)^T = B^T \cdot A^T$
$(AB)^{-1} = B^{-1}A^{-1}$

b) If a row (or column) in a sq. matrix A is a linear combination of the other rows (colums), then $|A| = 0$

① Function of two variables

Ex: $f(x,y) = 3x+y$ linear function $3x+y$ linear
$f(x,y) = x^2+y^2$ quadratic function x^2+y^2 quadratic
$f(x,y) = 2xy$ — " —
$f(x,y) = \frac{x+y}{x-y}$ rational function

In general: $f(x,y) = \{$ functional expression in x and y $\}$

Ex: $f(x,y) = x^2+y^2$

x	y	$z = f(x,y)$
0	0	$f(0,0) = 0$
1	0	$f(1,0) = 1$
0	1	$f(0,1) = 1$
⋮	⋮	⋮

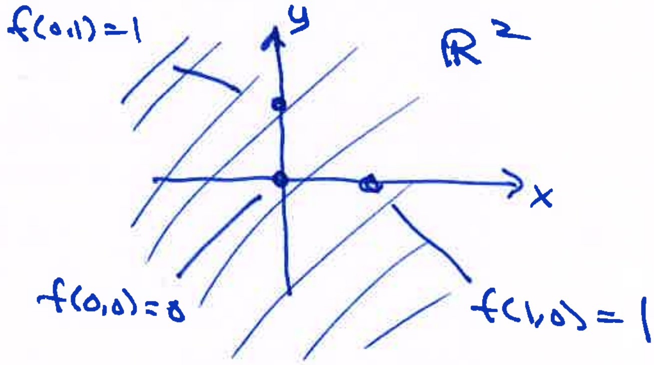
x, y: independent variables (input)
$z = f(x,y)$: dependent variable, (output)
function value

Domain of definition: D_f = all pairs (x,y) which we are allowed to put into the function f

Ex: $f(x,y) = x^2 + y^2$

$D_f = \mathbb{R}^2$

= all (x,y) in the two-dimensional coord. system

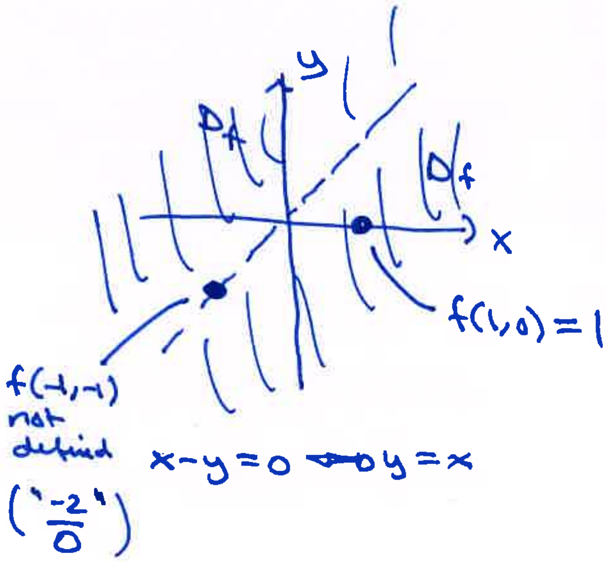


$f(x,y) = \frac{x+y}{x-y}$

D_f: $x - y \neq 0$

$D_f = \{(x,y) : x - y \neq 0\}$

(all pts (x,y) such that $x - y \neq 0$)



Range: $R_f = V_f$ = all z-values you can get when $z = f(x,y)$ for points (x,y) in D_f.

Ex: $f(x,y) = x^2 + y^2$, $R_f = [0, \to)$ $\iff$ $f_{min} = 0$, f_{max} does not exist (goes to ∞)

$f(x,y) = 3x + y$

$R_f = (-\infty, \infty) = \mathbb{R}$

(2) Graph of f:

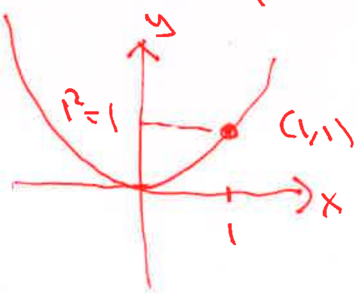
all points (x,y,z) such that $z = f(x,y)$ with (x,y) in D_f

When $f(x)$ is a fn. in one variable:

all pts (x,y) s.t. $y = f(x)$ and x is in D_f

Ex: $f(x) = x^2$

$y = x^2 \to$ all pts (x, x^2)



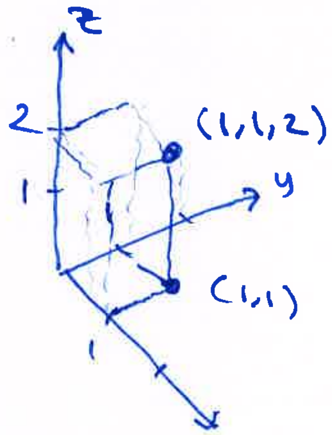
Geometrically, the graph of $f(x,y)$ is drawn in the xyz-coord. system.

Ex: $f(x,y) = x^2+y^2$

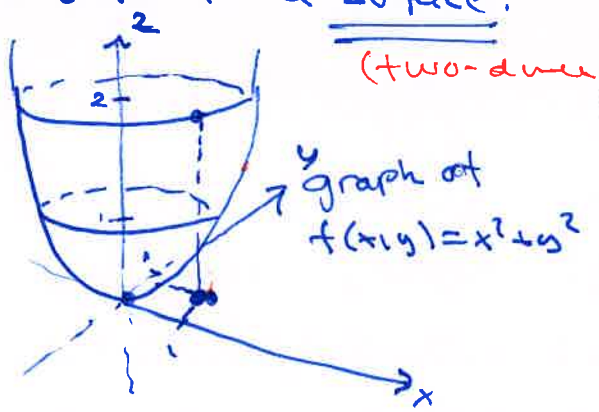
$f(1,1) = 2$

$\Updownarrow$

$(1,1,2) = (x,y,z)$



The graph is a surface. (two-dimensional)



Level curves: (horizontal cuts)

The curve $f(x,y) = c$ for a constant c.

= All pts (x,y,z) on the graph of f with $z = c$, i.e. all pts. on the graph at height c.

Ex: $f(x,y) = x^2+y^2$

$\Downarrow$

Level curve: $f(x,y) = c$

$x^2+y^2 = c$

$c=0$: $x^2+y^2 = 0$

$\to$ pt $(0,0)$.

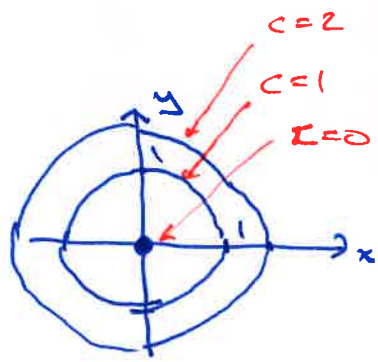
$c=1$: $x^2+y^2 = 1$

$\to$ circle $r=1$, centr $(0,0)$

$c=2$: $x^2+y^2 = 2$

$\to$ circle, $r=\sqrt{2}$

$c=-1$: $x^2+y^2 = -1 \to$ no pts.

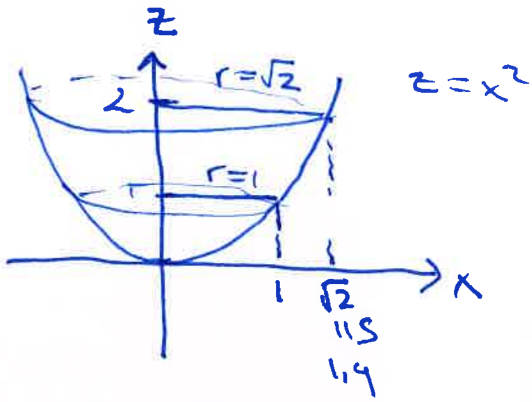


Vertical cuts:

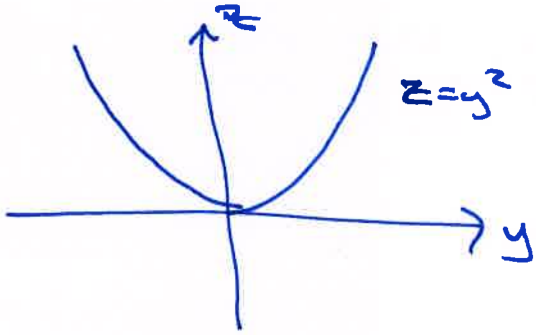
Ex $y = 0$

$z = f(x,y) = x^2 + y^2$

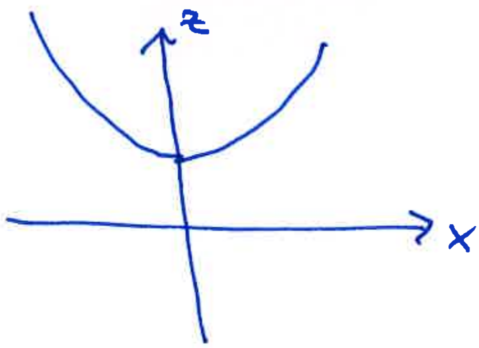
$z = x^2$



$x=0$: $z = y^2$



$y=1$: $z = x^2+1$



Ex: $f(x,y) = -x^2 - y^2$

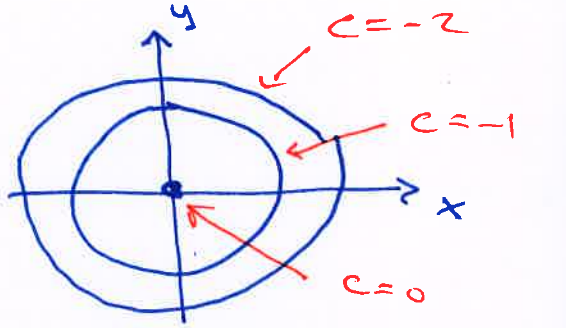
Level curves:

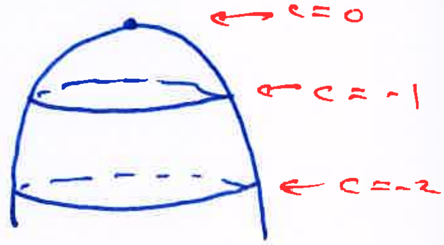
$c=0$: $-x^2-y^2=0$
$\to$ pt. $(0,0)$

$c=1$: $-x^2-y^2=1$
$x^2+y^2=-1 \to$ no pts

$c=-1$: $-x^2-y^2=-1$
$x^2+y^2=1 \to$ circle $r=1$

$c=-2$: $-x^2-y^2=-2$
$x^2+y^2=2 \to$ circle $r=\sqrt{2}$





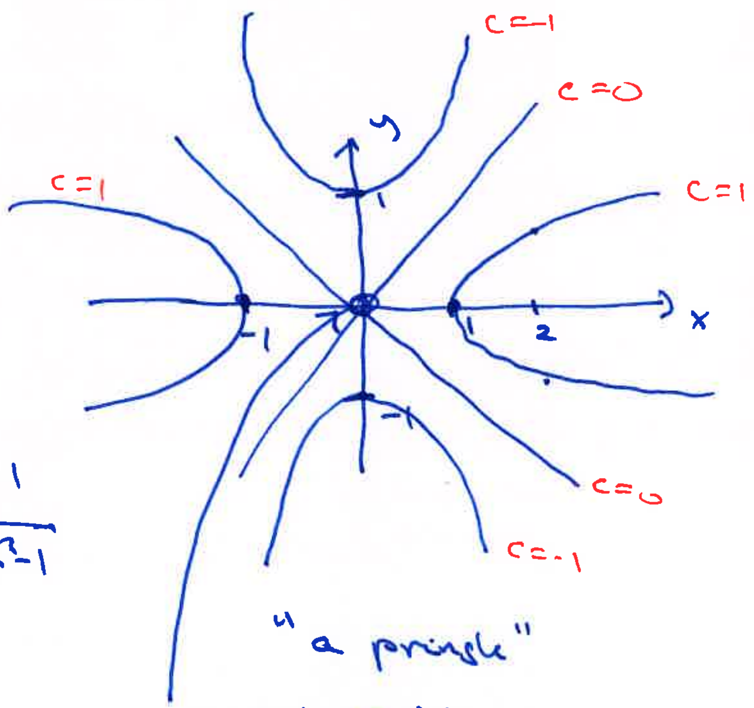
Ex: $f(x,y) = x^2 - y^2$

Level curves:

$c=0$: $x^2-y^2=0$
$(x+y)(x-y)=0$
$y=x$ or $y=-x$

$c=1$: $x^2-y^2=1 \to x^2=y^2+1$
$y^2=x^2-1 \to y=\pm\sqrt{x^2-1}$

$c=-1$: $x^2-y^2=-1$
$y^2=x^2+1 \to y=\pm\sqrt{x^2+1}$



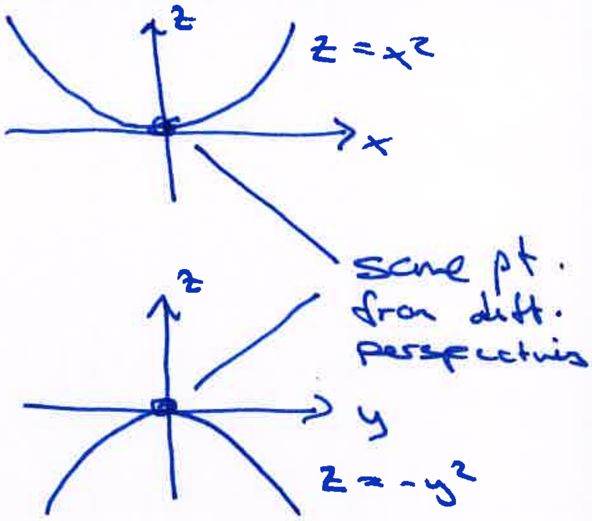
"a pringle"

saddle point: not max
not min

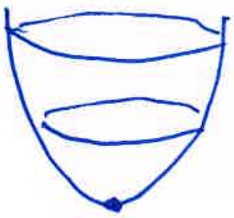
Cuts:

$y=0$
$z=x^2$

$x=0$
$z=-y^2$



$f(x,y)=x^2+y^2$



min

$f(x,y)=-x^2-y^2$

max

$f(x,y)=x^2-y^2$

"a pringle"

Saddle point
(not max, not min)

(3) Linear functions

$f(x,y) = ax + by + c$ linear fn.

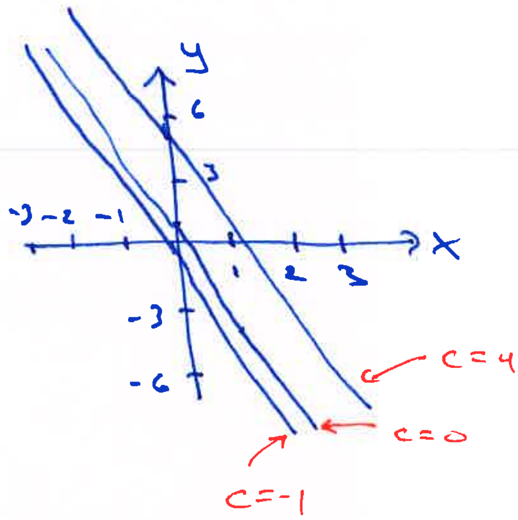
Ex: $f(x,y) = 3x + y - 1$

Level curves:

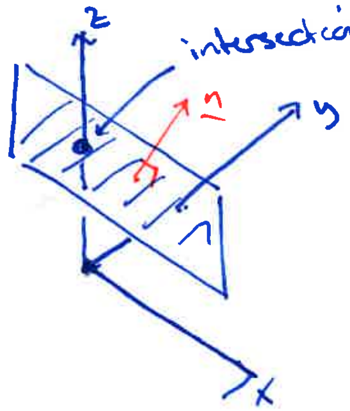
$c=0$: $3x+y-1=0$
$y = 1-3x$

$c=-1$: $3x+y-1=-1$
$y=-3x$

$c=4$: $3x+y-1=4$
$y=5-3x$



The graph of f is a plane $\Longleftrightarrow$ f is a linear function



intersection with z-axis

$f(0,0)=c$

$f(x,y) = ax + by + c$

$\underline{n} = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

a plane = a surface with no curvature ("straight")

Inner product / dot product of vectors:

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$$\underset{\text{dot prod.}}{\underline{v} \cdot \underline{w}} = 1 \cdot 4 + 2 \cdot (-3) = 4 - 6 = \underline{\underline{-2}} \quad \text{dot product}$$

$$= \underset{\substack{\uparrow \\ \text{matrix multiplication}}}{\underline{v}^T \cdot \underline{w}} = \underset{1 \times 2}{\begin{pmatrix} 1 & 2 \end{pmatrix}} \cdot \underset{2 \times 1}{\begin{pmatrix} 4 \\ -3 \end{pmatrix}} = \underline{\underline{-2}}$$

General defn:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} :$$

Defn of dot product:

$$\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + \dots + v_n \cdot w_n$$

$$\langle \underline{v}, \underline{w} \rangle = v_1 w_1 + \dots + v_n w_n$$

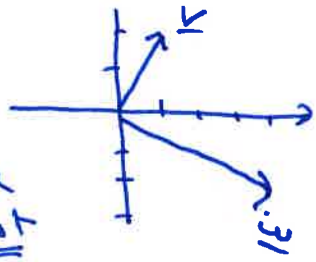
Result:

$$\boxed{\underline{v} \cdot \underline{w} = 0 \iff \underline{v} \perp \underline{w}}$$

($\underline{v}$ is normal to $\underline{w}$; angle is $90°$)

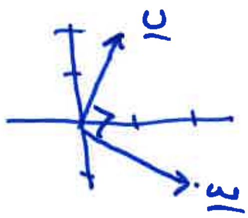
Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$$\underline{v} \cdot \underline{w} = -2 \neq 0 \Longrightarrow$$ angle is not 90°



Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left.\underline{v} \cdot \underline{w} = 1 \cdot 2 + 2(-1) = 0 \right\} \quad \underline{v} \perp \underline{w}$$



Ex: Which vectors are normal to $\underline{v} = \begin{pmatrix}1\\2\\1\end{pmatrix}$?

$\underline{w} = \begin{pmatrix}x\\y\\z\end{pmatrix}$: $\underline{v} \perp \underline{w} \Leftrightarrow \underline{v}\cdot\underline{w} = 0$

$1\cdot x + 2\cdot y + 1\cdot z = 0$

$\underline{x + 2y + z = 0}$

$(1 \quad 2 \quad 1 \mid 0)$

$\begin{cases} y, z \text{ free} \\ x = -2y - z \end{cases}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2y-z\\y\\z\end{pmatrix} = \begin{pmatrix}-2y\\y\\0\end{pmatrix} + \begin{pmatrix}-z\\0\\z\end{pmatrix} = y\cdot\begin{pmatrix}-2\\1\\0\end{pmatrix} + z\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Answer: The vectors normal to $\underline{v} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ are all linear combination of

$$\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

The plane spanned by $\begin{pmatrix}-2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ has normal vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$.

Graph of a linear function:

$f(x,y) = ax + by + c$ has graph that can be described geometrically:

- it is a plane
- the intersection with the z-axis is at c
- the normal vector is $\begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

$$z = ax + by$$
$$\Updownarrow$$
$$0 = ax + by - z$$
$$\Updownarrow$$
$$\boxed{\begin{pmatrix} a \\ b \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0}$$