Plan

1. Partial derivation and the Hessian
2. Tangents of level curves
3. Gradients and directional derivatives

Review:

$f(x,y)$: fn. in two variables

Graph of f: $z = f(x,y)$ surface in 3d coord. system (xyz-coord. sys.)

$f(x,y) = ax + by + c$ linear $\Leftrightarrow$ graph of f is a plane (no curvature)

Level curves:

$f(x,y) = c$ $\leftarrow$ all pts on the graph of f with height $z = c$

Linear fn: $z = f(x,y) = ax + by + c$ $\leftarrow$
* a plane
* intersect z-axis at $z = c$
* normal vector $\begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

c = 0: $z = ax + by$

$ax + by - z = 0$

$\begin{pmatrix} a \\ b \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

Dot product:

$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$: $\underline{u} \cdot \underline{v} = u_1 v_1 + \ldots + u_n v_n$

$\underline{u} \cdot \underline{v} = 0 \Leftrightarrow \underline{u} \perp \underline{v}$ (perpendicular, 90° angle)

Problems:

2. b) $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ which vectors are normal to $\underline{v}$.

$\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $\underline{v} \perp \underline{w}$

$\Updownarrow$

$\underline{v} \cdot \underline{w} = 0$

$(1 \;\; 0 \;\; 1 \mid 0)$ $\boxed{x + z = 0}$ $x = -z$, y free, z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

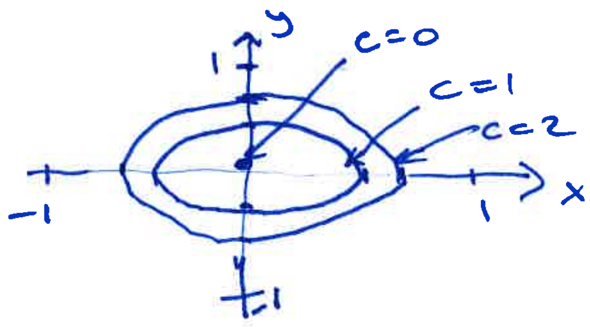
$$= y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

3. c) $f(x,y) = \boxed{4x^2 + 9y^2 = c}$ $c = -2, -1, 0, 1, 2$ level curves

$$\frac{4x^2}{c} + \frac{9y^2}{c} = 1$$

$$\frac{x^2}{c/4} + \frac{y^2}{c/9} = 1$$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ ellipse center $(0,0)$



$c > 0$: Ellipse center $(0,0)$
Half-axis:
$a = \sqrt{c}/2$, $b = \sqrt{c}/3$

$c = 0$: Pt. $(0,0)$

$c < 0$: No pts.

1 Partial derivatives

$f(x,y)$: fn. in two vars

Partial derivatives:

$$f'_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \begin{matrix} \leftarrow \Delta z \\ \leftarrow \Delta x \end{matrix} \quad = \frac{dz}{dx}$$

$$f'_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \quad \begin{matrix} \leftarrow \Delta z \\ \leftarrow \Delta y \end{matrix} \quad = \frac{dz}{dy}$$

How to compute partial derivatives:

- use derivation rules
- "same" derivation rules

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$$f'_x = 2x - 2 + 0 + 0 = \underline{2x-2}$$

$$f'_y = 0 - 0 + 2y + 4 = \underline{2y+4}$$

Ex: $f(x,y) = x^2 + xy + y^2$

$$f'_x = 2x + (xy)'_x + 0 = 2x + y \cdot (x)'_x$$

$$= 2x + y \cdot 1 = \underline{2x+y}$$

$$f'_y = 0 + x \cdot 1 + 2y = \underline{x+2y}$$

Ex: $f(x,y) = \dfrac{x+y}{x-y}$ $\quad x+y = u, \quad x-y = v$

$$f'_x = \frac{u'_x v - u v'_x}{v^2} = \frac{1\cdot(x-y)-(x+y)\cdot 1}{(x-y)^2} = \underline{\underline{\frac{-2y}{(x-y)^2}}}$$

$$f'_y = \frac{1\cdot(x-y)-(x+y)\cdot(-1)}{(x-y)^2} = \underline{\underline{\frac{2x}{(x-y)^2}}}$$

Ex: $f(x,y) = \sqrt{x^2 \cancel{+y^2}} = \sqrt{x^2+y^2} = \sqrt{u}, \quad u = x^2+y^2$

$$f'_x = \frac{1}{2\sqrt{u}}\cdot(x^2+y^2)'_x = \frac{1}{2\sqrt{u}}\cdot 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \underline{\underline{\frac{x}{\sqrt{x^2+y^2}}}}$$

$$f'_y = \frac{1}{2\sqrt{u}}\cdot(x^2+y^2)'_y = \frac{\cancel{2}y}{\cancel{2}\sqrt{u}} = \underline{\underline{\frac{y}{\sqrt{x^2+y^2}}}}$$

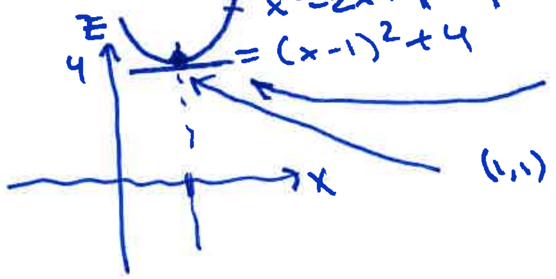
Interpretation of partial derivatives:

Ex: $f(x,y) = x^2-2x+y^2+4y$, $\quad (a,b) = (1,1)$

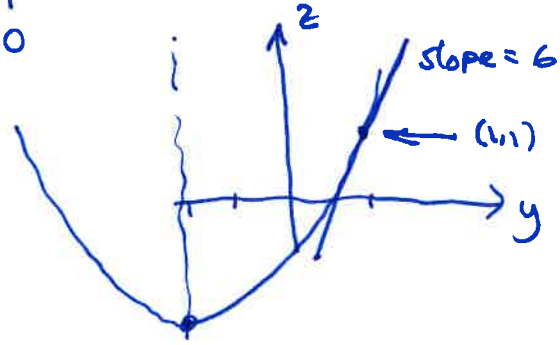
$f'_x(x,y) = 2x-2$ $\qquad f'_x(1,1) = \underline{0}$

$f'_y(x,y) = 2y+4$ $\qquad f'_y(1,1) = \underline{6}$

$\underline{y=1}$: $z = x^2-2x+5$
$= x^2-2x+1+4$
$= (x-1)^2+4$



$\underline{x=1}$: $z = -1+y^2+4y$
$= (y+2)^2-5$



$f'_x(a,b)$: slope of tangent of f in the x-direction
(we keep $y=b$ const. and change x)

$f'_y(a,b)$: slope of tangent of f in the y-direction
(we keep $x=a$ const and change y)

Defn.: A stationary pt of f is a point when $\boxed{f'_x = f'_y = 0}$ ← FOC = first order conditions

Ex: $f = x^2+4xy+y^2$

$$\boxed{\begin{aligned} f'_x &= 2x+4y = 0 \\ f'_y &= 4x+2y = 0 \end{aligned}}$$

FOC

$2x = -4y \Rightarrow x = -2y$

$4(-2y)+2y = 0$

$-6y = 0$

$y=0 \Rightarrow x=0$

Stationary pts: $(x,y) = (0,0)$

Max/min-problems:

(x,y) max/min for $f \Rightarrow (x,y)$ stationary pt. of f

The Hessian: $H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$f'_x = \underline{2x-2}$ $\quad f''_{xx} = 2 \quad f''_{xy} = 0$

$f'_y = \underline{2y+4}$ $\quad f''_{yx} = 0 \quad f''_{yy} = 2$

$f''_{xx} = (f'_x)'_x = (2x-2)'_x = 2 - 0 = 2$

$f''_{xy} = (f'_x)'_y = (2x-2)'_y = 0 - 0 = 0$

$f''_{yy} = (f'_y)'_y = (2y+4)'_y = 2 + 0 = 2$

$\Downarrow$

$H(f) = \underline{\underline{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}}$ Hessian matrix

Note: 1) $H(f)$ has entries which are functions in x and y

If (a,b) is a stationary pt, we can compute $H(f)(a,b) = \begin{pmatrix} f''_{xx}(a,b) & f''_{xy}(a,b) \\ f''_{yx}(a,b) & f''_{yy}(a,b) \end{pmatrix}$

2) $H(f)$ is symmetric; that is $f''_{xy} = f''_{yx}$.

(2) Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curves:

$f(x,y) = c$

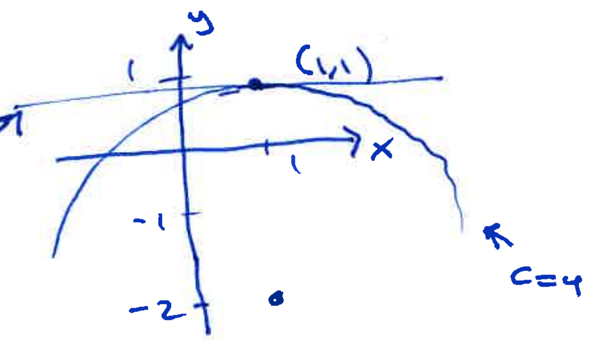
$\underline{x^2 - 2x} + \underline{y^2} + 4y = c$
$\quad +1 \qquad\quad +4 \qquad\qquad +1 \; +4$

$$\boxed{(x-1)^2 + (y+2)^2 = c+5}$$

Circles $c > -5$

center $(1,-2)$

radius $\sqrt{c+5}$



$f(1,1) = -1 + 5 = 4$

$\underline{c=4}$: $f(x,y) = 4 \leftarrow (1,1)$ lies on this level curve

Tangent to level curve $f(x,y) = c$ at (a,b):

$$y - b = y'(a,b) \cdot (x - a)$$

where $y'(a,b) = -\dfrac{f'_x(a,b)}{f'_y(a,b)}$ $\leftarrow$ $\boxed{y' = -\dfrac{f'_x}{f'_y}}$

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$f'_x = 2x - 2 \rightarrow f'_x(1,1) = \underline{0}$

$f'_y = 2y + 4 \rightarrow f'_y(1,1) = \underline{6}$

$\Bigg\}\; y'(1,1) = -\dfrac{0}{6} = 0$

Tangent: $y - 1 = 0 \cdot (x-1)$

$\underline{y = 1}$

Slope of the tangent line of a level curve of f:

$$y' = -\frac{f'_x}{f'_y}$$

Ex: $f(x,y) = x^2y - xy^2 = xy(x-y)$

Level curve: $x^2y - xy^2 = c$

Find y': a) $f'_x = 2xy - y^2$

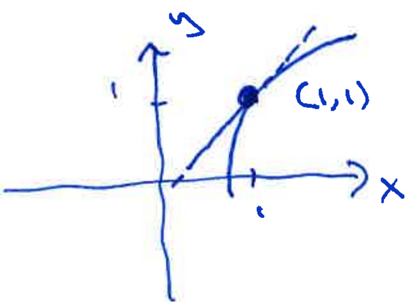
$f'_y = x^2 - 2xy$

$$\Downarrow$$

$$y' = -\frac{2xy-y^2}{x^2-2xy}$$

For instance:

$(1,1)$ is on $f(x,y)=0$: $y'(1,1) = -\frac{2-1}{1-2} = \underline{1}$



$\Rightarrow$ Tangent: $y-1 = 1\cdot(x-1)$

$y-1 = x-1$

$\underline{y=x}$

b) Implicit derivation:

$x^2y - xy^2 = c \quad | \; (\;)'_x$

$(2x\cdot y + x^2\cdot y') - (1\cdot y^2 + x\cdot 2y\cdot y') = 0$

$(2xy - y^2) + (x^2 - 2xy)\,y' = 0$

$f'_x + f'_y \cdot y' = 0 \qquad \frac{f'_y y'}{f'_y} = \frac{-f'_x}{f'_y}$

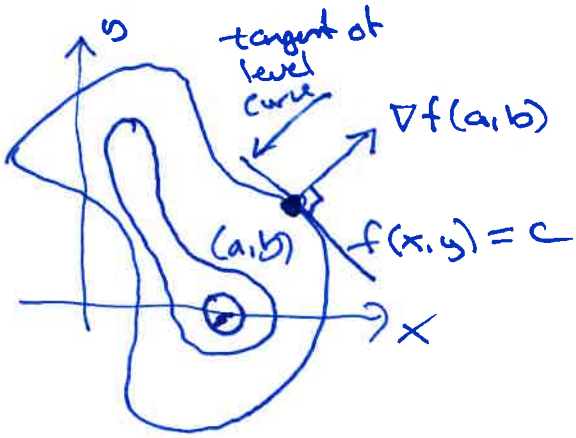
$$y' = -\frac{f'_x}{f'_y}$$

③ Gradients and directional derivatives

Gradient of f : $\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$

gradient of f

Picture:



* $\nabla f(a,b)$ is drawn as a vector starting at (a,b)
* $\nabla f(a,b)$ is normal to the tangent of the level curve at (a,b).
* the direction of the gradient is where f increases fastest.

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} 2x-2 \\ 2y+4 \end{pmatrix}$$

$$\nabla f(1,1) = \begin{pmatrix} 2\cdot 1 - 2 \\ 2\cdot 1 + 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$f(1,1) = 4$

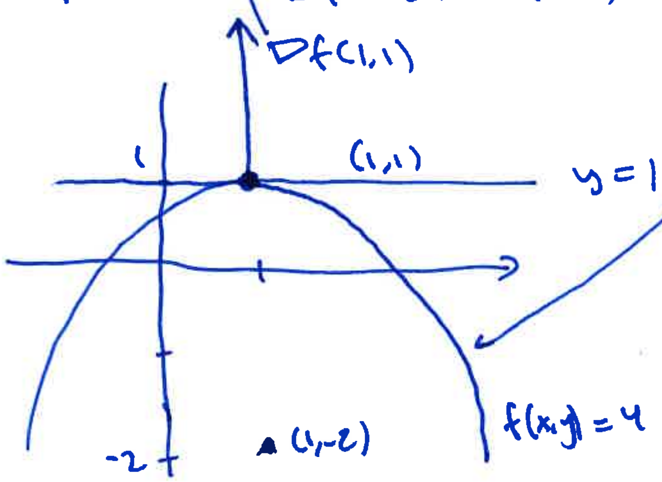
$f(x,y) = 4$

$$\underset{+1}{x^2 - 2x} + \underset{+4}{y^2 + 4y} = \underset{+5}{4}$$

$$(x-1)^2 + (y+2)^2 = 9$$

center: $(1,-2)$

radius: 3



Directional derivative:

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$: $f'_{\underline{a}} = a_1 \cdot f'_x + a_2 \cdot f'_y = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \underline{a} \cdot \nabla f$

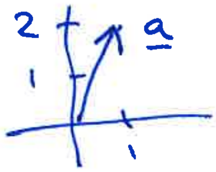
marginal change in $z = f(x,y)$ when you move in the direction of $\underline{a}$ in the xy-plane

$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $f'_{\underline{a}} = f'_x$

$a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $f'_{\underline{a}} = f'_y$

special cases

$\underline{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: $f'_{\underline{a}} = 1 \cdot f'_x + 2 \cdot f'_y$



marginal change in $z = f(x,y)$ when you move (x,y) in the direction of $\underline{a}$.