Lecture 28   EBA 2910

Plan:

① Stationary pts and second derivative test

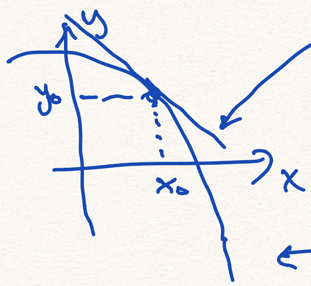
② Max and min problem

Review:

a) <u>Hessian matrix:</u> $H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$

<u>Fact:</u> $H(f)$ is a symmetric matrix ($f''_{xy} = f''_{yx}$)

b) <u>Level curves:</u>



Tangent at the level curve $f(x,y) = c$ at (x_0, y_0).

$\leftarrow f(x,y) = c$

<u>Equation:</u>

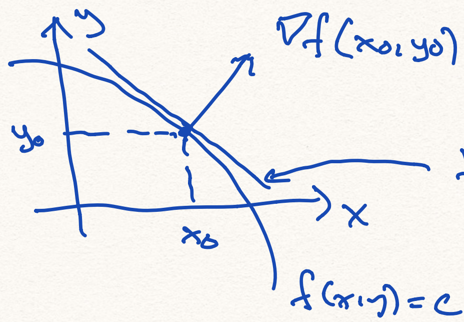
$y - y_0 = a \cdot (x - x_0)$

where a is the slope of the tangent

Slope of tangent: of a level curve $f(x,y)=c$

$$a = y'(x_0,y_0) = -\frac{f'_x(x_0,y_0)}{f'_y(x_0,y_0)}$$

Gradient of f:



gradient of f at (x_0,y_0)

tangent of the level curve

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \quad : \quad \nabla f(x_0,y_0) = \begin{pmatrix} f'_x(x_0,y_0) \\ f'_y(x_0,y_0) \end{pmatrix}$$

Note: i) The gradient is normal to the tangent of the level curve

ii) The gradient pts in the direction where f will increase the fastest

Directional derivative:

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\nearrow \underline{a}$

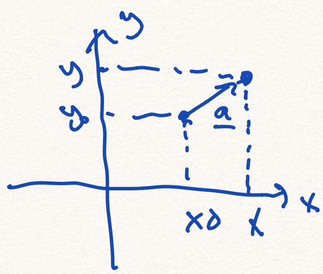
$$f'_{\underline{a}} = \underline{a} \cdot \nabla f = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$$

$$= \underline{a_1 \cdot f'_x + a_2 \cdot f'_y}$$

$f'_{\underline{a}}$ gives the marginal change in $f(x,y)$ when we move in the direction of $\underline{a}$.

Special cases:
$$\begin{cases} \underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad f'_{\underline{a}} = f'_x \\ \rightarrow \\ \underline{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad f'_{\underline{a}} = f'_y \\ \uparrow \end{cases}$$

Linear approximation of $f(x,y)$ close to (x_0,y_0)



$$\underline{a} = \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$f'_{\underline{a}} = \underline{a} \cdot \nabla f(x_0,y_0)$$
$$= (x-x_0)\cdot f'_x(x_0,y_0) + (y-y_0)\cdot f'_y(x_0,y_0)$$

$f(x_0,y_0)$ at (x_0,y_0)

$$\boxed{f(x,y) \cong f(x_0,y_0) + f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0)}$$

In one variable:

$$f(x) \simeq f(x_0) + f'(x_0)\cdot(x-x_0)$$

Linear approx. of f close to $x=x_0$ (tangent line)

Ex: $f(x,y) = x^2 + y^3 + y^2$ $\qquad (x_0,y_0) = (1,1)$

$f(1,1) = \underline{3}$

$f'_x = 2x \qquad f'_x(1,1) = \underline{2}$

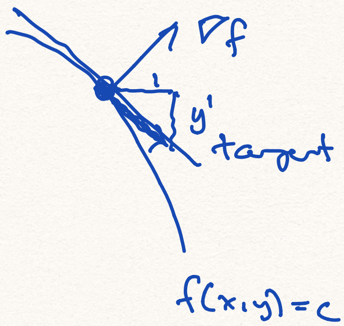
$f'_y = 3y^2+2y \qquad f'_y(1,1) = \underline{5}$

Lin. approx. of f close to $(1,1)$:

$$f(x,y) \simeq 3 + 2(x-1) + 5(y-1)$$

$$f(1.2, 0.9) \cong 3 + 0.4 - 0.5 = \underline{\underline{2.9}}$$

Problem set 27, Pb. 6



i) ∇f normal to the tangent

ii) f increases along ∇f.

Explanation:

i) $\begin{pmatrix} 1 \\ -f'_x / f'_y \end{pmatrix}$ is a vector along the tangent

$$\nabla f \cdot \begin{pmatrix} 1 \\ -f'_x / f'_y \end{pmatrix} = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -f'_x / f'_y \end{pmatrix}$$

$$= f'_x \cdot 1 + \cancel{f'_y} \cdot \left(- f'_x / \cancel{f'_y}\right) = 0 \quad ✓$$

ii) $\underline{a} = \nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$

$$f'_{\underline{a}} = \underline{a} \cdot \nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \cdot \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$$

$$= (f'_x)^2 + (f'_y)^2 \geq 0 \quad ✓$$

① Stationary pts and second derivative test

Problem: $\boxed{\max/\min\ f(x,y)}$

optimization (no constraints)

a) Stationary points of f:

Defn: A stationary pt. of f is a pt. such that

$$\boxed{\begin{aligned} f'_x &= 0 \\ f'_y &= 0 \end{aligned}}$$

FOC (first order conditions)

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Stat. pts:

$$\left.\begin{aligned} f'_x &= 2x - 2 = 0 \quad & x = 1 \\ f'_y &= 2y + 4 = 0 \quad & y = -2 \end{aligned}\right\} \quad (x,y) = \underline{(1,-2)}$$

Ex: $f(x,y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f'_x &= 3x^2 - 3y = 0 \\ f'_y &= -3x + 3y^2 = 0 \end{aligned}\right\} \quad \begin{aligned} x^2 &= y \\ y^2 &= x \end{aligned}$$

$$x^2 = y \qquad \Rightarrow \quad y = x^2$$
$$y^2 = x \qquad\qquad (x^2)^2 = x$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0$$
$$\underline{x = 0} \quad \text{or} \quad x^3 = 1$$
$$x = \sqrt[3]{1} = \underline{1}$$
$$\underline{y = 0} \qquad\qquad \underline{y = 1}$$

Stat. pts:

$(x,y) = (0,0)$,
$(1,1)$

Result: If (x,y) is a max/min for f, then we have one of the following:

i) (x,y) is a stationary pt. of f

ii) (x,y) is a pt. where f'_x or f'_y is not defined (critical pt)

iii) (x,y) is a boundary pt. in D_f.

Candidate pts: stationary pt or a pt that satisfies ii) or iii) above.

Ex: $f = x^2 - 2x + y^2 + 4y$, $D_f = \mathbb{R}^2$

Stat. pts: $(1,-2)$

no boundary pts, no pts where f'_x / f'_y is not defined

$f'_x = 2x - 2$

$f'_y = 2y + 4$

$\Rightarrow$ Candidate pts: $\boxed{(1,-2)}$ $f(1,-2) = \underline{-5}$

Ex: $f = x^3 - 3xy + y^3$, $D_f = \mathbb{R}^2$

Stat pts: $(0,0), (1,1)$ (no pts of type ii) or iii))

$\Rightarrow$ Cand. pts: $\boxed{(0,0),\ (1,1)}$

$f(0,0) = 0$ $f(1,1) = -1$

Classification of stationary pts:

Defn: A stationary pt (x^*, y^*) is called

i) a local min if $f(x^*, y^*) \le f(x,y)$ for all pts (x,y) close to (x^*, y^*)

ii) a local max if $f(x^*, y^*) \ge f(x,y)$ for all pts (x,y) close to (x^*, y^*)

iii) a saddle pt otherwise

Second derivative test:

Let (x^*, y^*) be a stationary pt. of f. Consider

$$H(f)(x^*,y^*) = \begin{pmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{yx}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

and compute

$$\det H(f)(x^*,y^*) = AC - B^2$$

$$\operatorname{tr} H(f)(x^*,y^*) = A + C$$

We have:

i) $AC - B^2 > 0$, $A + C > 0 \Rightarrow (x^*, y^*)$ local min

ii) $AC - B^2 > 0$, $A + C < 0 \Rightarrow (x^*, y^*)$ local max

iii) $AC - B^2 < 0 \Rightarrow (x^*, y^*)$ saddle pt

If $AC - B^2 = 0$, the test is inconclusive.

(We must use other methods to find out what kind of point we have, usually the definition of local max/min)

Ex: $f = x^2 - 2x + y^2 + 4y$

$f'_x = 2x - 2$

$f'_y = 2y + 4$

$$H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Stat. pts: $(1,-2)$

$$H(f)(1,-2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det = 4 - 0 = 4 > 0$

$\operatorname{tr} = 2+2 = 4 > 0$

$\Downarrow$

$(1,-2)$ is local min

Ex: $f = x^3 - 3xy + y^3$

$$\left.\begin{array}{l} f'_x = 3x^2 - 3y \\ f'_y = -3x + 3y^2 \end{array}\right\} \quad H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Stat pts: $(0,0)$, $(1,1)$

$$H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

$\det = 0 - 9 = -9$

$(0,0)$ saddle pt

$$H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$\det = 36 - 9 = 27 > 0$

$\operatorname{tr} = 6+6 = 12 > 0$

$(1,1)$ is local min

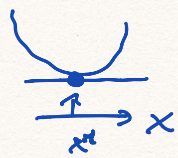
<u>Explanation:</u> i) $\det = AC - B^2 > 0$

$$AC > B^2 \geq 0$$

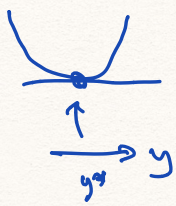
$$AC > 0$$

$A > 0, C > 0$ or $A < 0, C < 0$

$A = f''_{xx}(x^*, y^*) > 0$

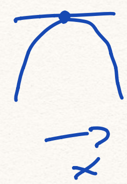


$C = f''_{yy}(x^*, y^*) > 0$



local min

$A + C > 0$



local max

$A + C < 0$

2 Global max / min

$$\boxed{\max/\min\ f(x,y)}$$ unconstrained optimization problem

Method: to find candidate pts, and classify them locally

a) Find all stationary pts of f, by solving $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$. Add possible pts that satisfy ii) or iii), that is, pts where f'_x / f'_y are not defined, and boundary pts.

$\Rightarrow$ List of candidate pts.

b) Classify all stationary pts as local max / local min / saddle pts using second derivative test. This will not work:

i) If $AC-B^2=0$ for a stationary pt.

ii) If the pt is not stationary

To find global max/min = max/min:

global max $\Rightarrow$ local max
global min $\Rightarrow$ local min

Ex: $f = x^3 - 3xy + y^3$

$(0,0)$ saddle pt
$(1,1)$ local min
} cand. pts.

No max: no local max

Min: $f(1,1) = -1$ $\leftarrow$ Is this the smallest value of f?

$x=1$: $f(1,y) = 1 - 3y + y^3$

$y \to -\infty \Rightarrow f(1,y) \to -\infty$

no global min

$$\left(\begin{aligned} f(1,-3) &= 1 + 9 - 27 \\ &= -17 < -1 \end{aligned} \right)$$