Plan:

1. Optimization without constraints
2. Extreme value theorem
3. Constrained optimization

1. Unconstrained optimization

max/min $f(x,y)$

- no extra conditions on (x,y) = unconstrained

Ex: max x^2+y^2 when $xy=1$ ← constraint

Method:

1. Find all candidate pts

i) Stationary pts: $\underbrace{f'_x = f'_y = 0}_{\text{FOC}}$

ii) Pts where f'_x or f'_y are not defined (critical pts)

iii) Boundary pts of D_f.

② Classify stationary pts

- local max
- local min
- saddle pts

Second derivative test:

(x^*, y^*) stationary pt

$$H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det = AC - B^2$

$\mathrm{tr} = A + C$

$AC - B^2 > 0,\ A + C > 0 \Rightarrow (x^*, y^*)$ local min

$AC - B^2 > 0,\ A + C < 0$ " local max

$AC - B^2 < 0$ " saddle pt.

Problems:

i) Stationary pt with $AC - B^2 = 0$

ii) Candidate pts that are not stationary pts

use other methods (defn.)

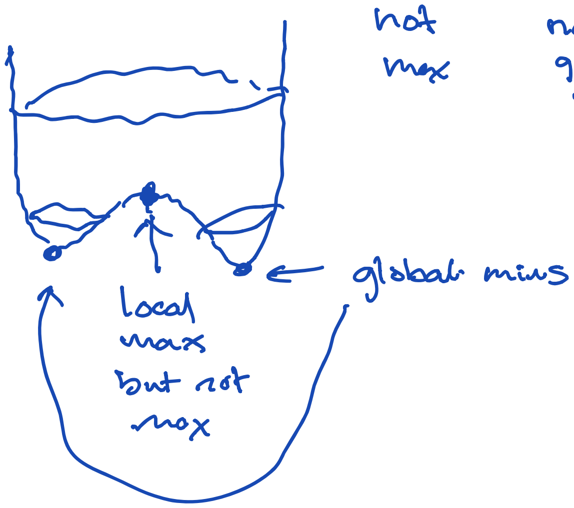
③ Determine if local max/min are global max/min

global max $\Rightarrow$ local max

global min $\Rightarrow$ local min

Examples: One variable

Two variables



How to determine:

Ex: $f(x,y) = \underline{x^3 - 3xy + y^3}$

Cand. pts:	
$(0,0)$	saddle pt.
$(1,1)$	local min

$f(0,0) = 0 \quad f(1,1) = -1$

No global max (no local max)

Global min: $f(1,1) = -1$ is a candidate (the only candidate)

Try $y=0$: $f(x,0) = x^3$

$f(-2,0) = (-2)^3 = -8 < -1$ $\quad$ **no global min**

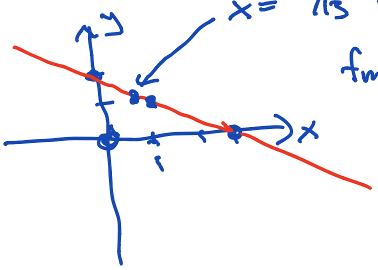
② Constrained optimization

$$\max/\min \underbrace{f(x,y)}_{\text{objective function}} \quad \text{when} \quad \underbrace{\left\{\begin{array}{l} g(x,y)=a \\ \text{or} \\ g(x,y)\le a \\ \text{or} \\ \text{similar} \end{array}\right.}_{\text{constraints}}$$

D = the set of all points (x,y) that satisfy all constraints

admissible points

Ex: min $f(x,y)=x^2+y^2$ when $\underbrace{\boxed{2x+3y \overset{?}{=} 6}}_{\substack{\text{admissible pts} \\ = \text{ a line} \\ \text{(red line)}}}$



$x = 12/13$ is the min. pt.

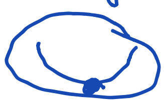
$$f_{min} = \frac{13}{9}\cdot\left(\frac{12}{13}\right)^2 - \frac{8}{3}\cdot\left(\frac{12}{13}\right) + 4 = \underline{36/13}$$

$$\frac{3y}{3} = \frac{6-2x}{3}$$

$$y = 2 - \frac{2}{3}x$$

$$f\left(x, 2-\frac{2}{3}x\right) = x^2 + \left(2-\frac{2}{3}x\right)^2$$

$$= x^2 + 4 - \frac{8}{3}x + \frac{4}{9}x^2 = \frac{13}{9}x^2 - \frac{8}{3}x + 4$$



$$' = \frac{26}{9}x - \frac{8}{3} = 0 \quad 1\cdot 9 \quad 26x - 24 = 0 \quad x = \frac{24}{26} = \frac{12}{13}$$

First method: If the constraint is an equation that is easy to solve, you can solve it for x or y, and substitute it into the obj. fn. $f(x,y)$, and solve as an unconstrained problem in one variable.

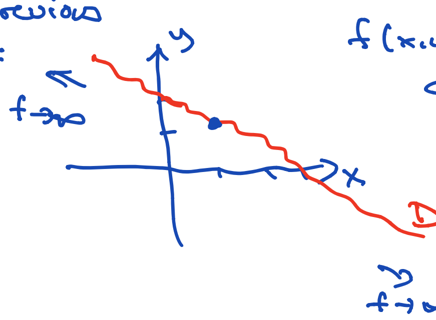
Extreme Value Thm: (EVT)

If f is a continuous function on a compact subset D of $\mathbb{R}^2$ (xy-plane), then f has a max and a min.

Use: $f = f(x,y)$ objective function

$D =$ the set of admissible pts
$=$ the set of all pts satisfying the constraints

In the previous example:



$f(x,y) = x^2+y^2$
cont. ✓

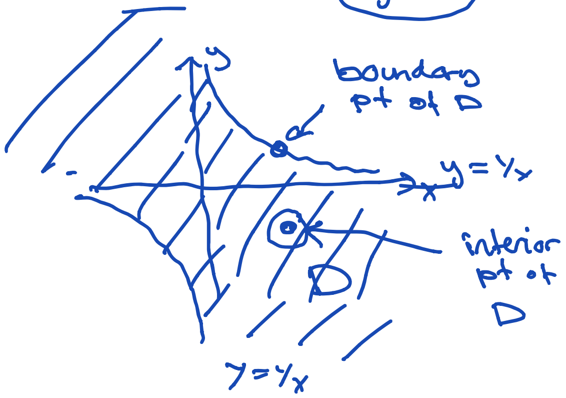
compact? No.

Defn: A set D is compact if it is closed and bounded.

i) Closed sets: If the constraints are given by $=$ (eqn.) or $\leq, \geq$ (closed inequalities) then the set D of adm. pts is closed.

Ex: $2x+3y=6$ is closed

Ex: D: $xy \leq 1$ ← D is closed ($\leq$)



$xy=1$: $y=1/x$ hyperbola
these pts are called boundary pts

$xy<1$:

$y<1/x$ $(x>0)$

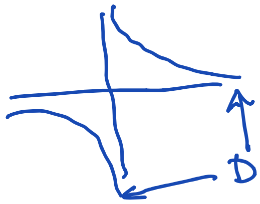
$y>1/x$ $(x<0)$

these pts are called interior pts of D

Closed sets = all boundary pts of D are included in D

Ex: D: $y = 1/x$

closed

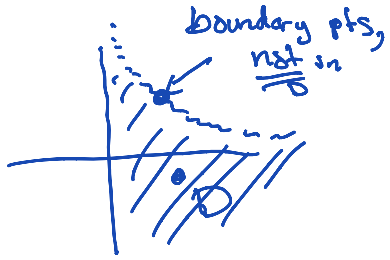


D has only boundary pts (no interior pts)

D: $xy < 1$, $x > 0$

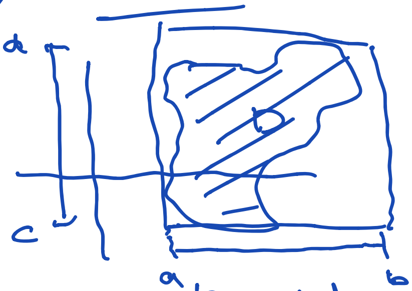
$x > 0$, $y < 1/x$

not closed



D: only interior pts

ii) Bounded



bounded

not bounded

D is bounded if there are finite numbers a, b, c, d such that

$$a \le x \le b$$
$$c \le y \le d$$

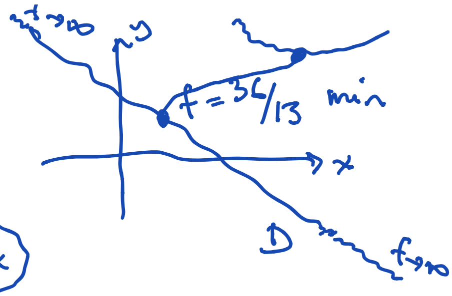
for all pts (x, y) in D.

Ex: D: $2x+3y=6$

closed, not bounded

$\Rightarrow$ not compact

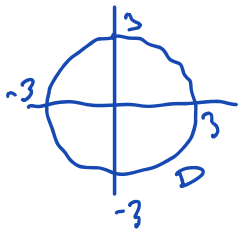
min
not max



D: $x^2+y^2=9$

closed, bounded $\Rightarrow$ compact

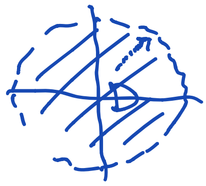
$$\left(\begin{array}{l} -3 \le x \le 3 \\ -3 \le y \le 3 \end{array}\right)$$



D: $x^2+y^2<9$

not closed, bounded

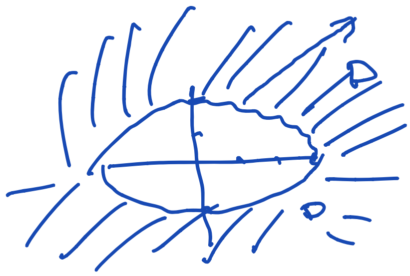
not compact



D: $4x^2+9y^2 \geq 36$

closed, not bounded

not compact.



$4x^2+9y^2=36$ $\quad$ $\div 36$

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

How do you find candidate pts for a problem with constraints?

A) Solve the inequality for x or y, and substitute in $f(x,y)$

$\Rightarrow$ optimization problem in one variable without constraints

B) Consider all possible candidate pts

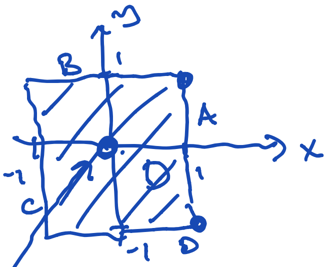
- all interior pts that are stationary
- all boundary pts in D

I will do an example below.

C) Lagrange's method. (next lecture)

An example of method B

max/min $f(x,y) = e^{xy}$ when $-1 \le x,y \le 1$.



Candidate pts:

a) Stationary pts of f that are interior pts in D:

$$f'_x = e^{xy} \cdot y = 0$$
$$f'_y = e^{xy} \cdot x = 0$$

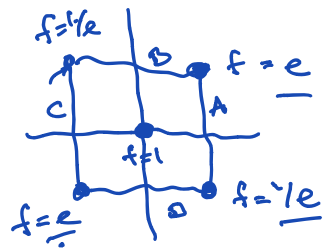
$\Downarrow$

$y = 0,\ x = 0 \Rightarrow (x,y) = (0,0)$ interior pt ✓

$\Downarrow$

$(x,y) = (0,0) \cancel{\to f(0,0) = 1}$

D is closed and bounded
$\Downarrow$
D compact
$\Downarrow$
there is a max and a min



b) All boundary pts:

A: $x = 1,\ -1 \le y \le 1$

$f(1,y) = e^y$

increasing function

$(e^y)' = e^y > 0$

Smallest value: on A

$f(1,-1) = e^{-1} = 1/e \simeq 0.37$.

Biggest value: on A

$f(1,1) = e^1 = e \simeq 2.72$

B: $y = 1,\ -1 \le x \le 1$

$f(x,1) = e^x$ → same biggest/smallest value by symmetry

C: $x = -1, \; -1 \le y \le 1$

$f(-1, y) = e^{-y}$

$(e^{-y})' = -e^{-y} < 0$

decreasing.

Biggest value on C

$f(-1,-1) = e \simeq 2{,}72$

Smallest value on C

$f(-1, 1) = 1/e \simeq 0.37$

D: Same as C, by symmetry

$f_{max} = e \simeq 2{,}72$ at $(x,y) = (1,1), (-1,-1)$

$f_{min} = \frac{1}{e} \simeq 0{,}37$ at $(x,y) = (1,-1), (-1,1)$