Lecture 30

EBA 2910

Plan:

1. Lagrange's method
2. Degenerate constraint
3. Global maximum / minimum

Review:

$$\boxed{\begin{array}{l}\textbf{Extreme Value Theorem}\\ \text{If } f \text{ is a continuous function on a compact}\\ \text{(closed and bounded) set } D\text{, then } f \text{ has a}\\ \text{maximum and minimum in } D\end{array}}$$

closed: if defined by
$=$, $\leq$, $\geq$

bounded: if there is a (finite) rectangle that contains all pts in D

f: objective fn.

D: set of adm. pts (pts that satisfy all constraints)

Important curves:

Ex:

i) Straight line: $ax+by=c$ $\qquad$ $2x+3y=12$

ii) Circle: $(x-x_0)^2+(y-y_0)^2=r^2$ $\qquad$ $x^2+y^2=9$

iii) Ellipse: $\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}=1$ $\qquad$ $4x^2+9y^2=36$

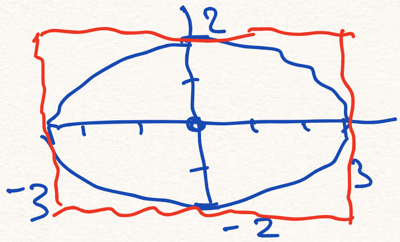
iv) Parabola: $y=a(x-x_0)^2+b$ $\qquad$ $y=4x^2-3$

v) Hyperbola: $(x-x_0)(y-y_0)=c$ $\qquad$ $xy=4$

Example:

$4x^2+9y^2=36$ $\quad$ $1:36$

$\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$



ellipse

$\boxed{\text{Compact set}} \Leftarrow \begin{cases} \checkmark \text{ closed} & (\Rightarrow) \\ \checkmark \text{ bounded} \end{cases}$

$-3 \le x \le 3$

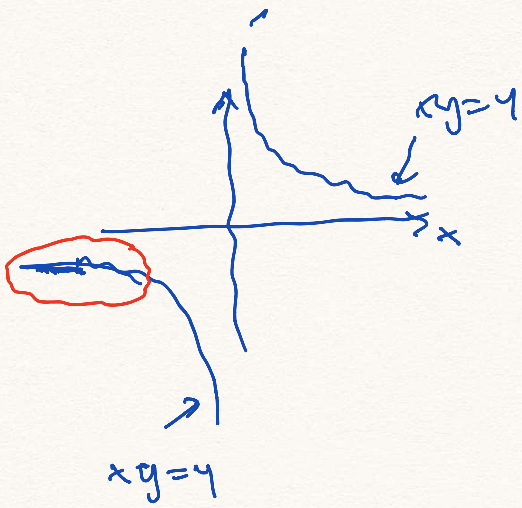
$-2 \le y \le 2$

Recall:

If D is closed and bounded, then the optimization problem has max/min. (EVT)

$\Rightarrow$ $\boxed{\text{one of the candidate pts is max/min}}$

Example:



$xy = 4$

$y = 4/x$

closed (=)

not bounded (for example, there is no lower bound on x)

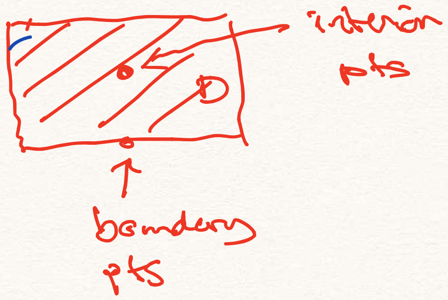
Review : Methods

i) Solve constraint for x or y, and substitute into $f(x,y)$, the objective fn.

$$\left.\begin{array}{l} \min \underbrace{x^2+y^2}_{f(x,y)} \text{ s.t. } xy=1 \\ \qquad\qquad\qquad\quad \Downarrow \\ \qquad\qquad\qquad\quad y = 1/x \end{array}\right\} \min f(x, 1/x) = x^2 + 1/x^2$$

Applies when the constraints are simple eqn.

ii) Run through all boundary pts



i) find stationary pts of f, and check if they are interior pts.

ii) check all pts on the boundary

Applies when the boundary is simple (straight lines)

Recall:

A max/min is either

i) an interior pt that is stationary ($f'_x = f'_y = 0$) or where f'_x / f'_y does not exist.

ii) a boundary pt

① Lagrange's method:

Applies to all cases with equality constraints, in particular when the equation is complicated.

Defn:

Lagrange problem = constrained optimization problem with equality constraints

In our case: $\boxed{\max/\min\ f(x,y) \text{ when } g(x,y) = a}$

Method: (Lagrange's method)

i) Write down the Lagrangian

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

λ: Lagrange multiplier (help variable)
(lambda)

Note: You can write $g(x,y) - a = 0$ and
$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda \cdot (g(x,y) - a)$

ii) Write down the necessary conditions:

FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ $\qquad f'_x - \lambda \cdot g'_x = 0$

$\qquad\qquad\qquad\qquad\quad f'_y - \lambda \cdot g'_y = 0$

↑ first order conditions

C: $\{ g(x,y) = a$

↑ constraint

Lagrange conditions: (necessary conditions)

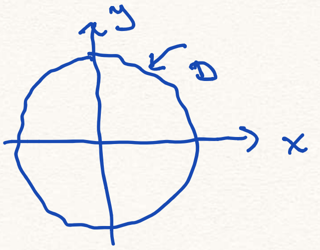
$$\boxed{\begin{array}{l} \text{FOC} \begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases} \\ \text{C} \quad g(x,y) = a \end{array}}$$

3 eqn's in 3 variables $(x, y; \lambda)$

iii) Solve FOC + C $\Rightarrow$ $\boxed{\text{ordinary candidate points}}$

Example: max/min $\underbrace{f(x,y)=4x-3y}_{\text{obj. fn.}}$ when $x^2+y^2=25$

$$\underbrace{g(x,y) = a}_{\text{constraint set } D}$$



D: circle, center (0,0), r = 5

D is compact $\begin{cases} \text{closed: } = \\ \text{bounded: circle} \end{cases}$

⇓ EVT

there is a max and a min

Lagrange's method:

$$\mathcal{L} = \underbrace{4x-3y}_{f(x,y)} - \lambda\cdot(\underbrace{x^2+y^2}_{g(x,y)})$$

$$\begin{array}{ll} \text{FOC} \begin{cases} \mathcal{L}'_x = \\ \mathcal{L}'_y = \end{cases} & \boxed{\begin{array}{ll} 4 - \lambda\cdot 2x & = 0 \\ -3 - \lambda\cdot 2y & = 0 \\ x^2 + y^2 & = 25 \end{array}} \\ \text{C} & \end{array}$$

FOC + C

We must solve this 3x3 system of equations

Note: If $\mathcal{L} = 4x-3y-\lambda(x^2+y^2-25)$, then

$$\mathcal{L}'_\lambda = -1\cdot(x^2+y^2-25) = 0 \quad | \cdot(-1)$$

$$x^2+y^2 = 25 \longleftarrow C$$

$$\boxed{\begin{array}{ll} (1) & 4-\lambda\cdot 2x=0 \\ (2) & -3-\lambda\cdot 2y=0 \\ (3) & x^2+y^2=25 \end{array}}$$

FOC + C

Could $\lambda=0$?

(1) $4-0\cdot 2x=0$

$4=0$

impossible

$\lambda \neq 0$

(1) $\frac{4}{2\lambda}=\frac{\cancel{2\lambda}x}{\cancel{2\lambda}}$ $\quad |:2\lambda$

$$x=\frac{4}{2\lambda}$$

(2) $\frac{-3}{2\lambda}=\frac{\cancel{2\lambda}y}{\cancel{2\lambda}}$ $\quad |:2\lambda$

$$y=\frac{-3}{2\lambda}$$

Ordinary candidate pts: FOC + C

$(x,y;\lambda)=(4,-3;\tfrac{1}{2}),$
$(-4,3;-\tfrac{1}{2})$

(3) $x^2+y^2=25$

$$\left(\frac{4}{2\lambda}\right)^2+\left(\frac{-3}{2\lambda}\right)^2=25$$

$$\frac{16}{4\lambda^2}+\frac{9}{4\lambda^2}=25$$

$$\frac{25}{4\lambda^2}=25 \quad |\cdot 4\lambda^2$$

$$\frac{25}{100}=\frac{\cancel{25}\,4\lambda^2}{\cancel{100}} \quad :100$$

$$\lambda^2=\tfrac{1}{4}$$

$$\lambda=\pm\sqrt{\tfrac{1}{4}}$$

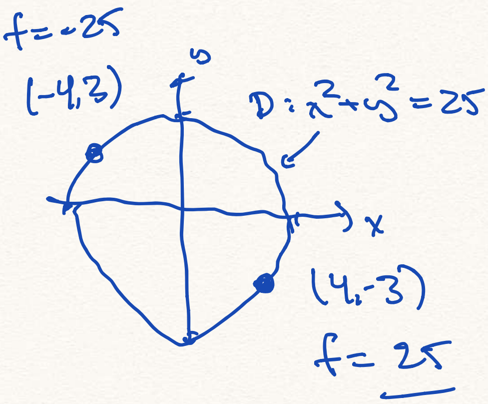
$$=\pm\tfrac{1}{2}$$

Conclusions: max/min $f(x,y) = 4x-3y$ when $x^2+y^2=25$

D is compact $\overset{\text{EVT}}{\Longrightarrow}$ there is a max/min.

Candidate pts: $(x,y;\lambda) = (4,-3;\tfrac{1}{2})$ $f=25$

$(-4,3;-\tfrac{1}{2})$ $f=-25$



$f_{max} = \underline{\underline{25}}$ at $(4,-3)$

$f_{min} = \underline{\underline{-25}}$ at $(-4,3)$

(since D is compact)

Result:

If (x^*, y^*) is a maximum or minimum pt. in a Lagrange problem, then one of the following conditions hold:

i) there is a λ such that (x^*, y^*, λ) satisfy FOC+C (Lagrange conditions)

ii) the point (x^*, y^*) is an admissible pt with degenerate constraint, i.e such that

$$\boxed{\nabla g(x^*, y^*) = 0 \Leftarrow \begin{cases} g'_x(x^*, y^*) = 0 \\ g'_y(x^*, y^*) = 0 \end{cases} \quad \text{and} \quad g(x^*, y^*) = a}$$

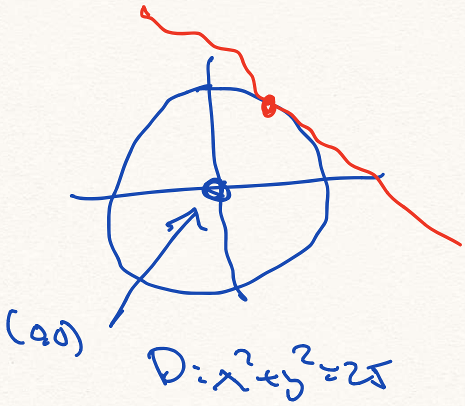
case i): Ordinary candidate pts.

FOC+C $\boxed{\begin{array}{ll} \mathcal{L}'_x = 0 & g(x,y) = a \\ \mathcal{L}'_y = 0 & \end{array}}$

case ii): exceptional candidate pts

$\boxed{\begin{array}{ll} g'_x = 0 & g(x,y) = a \\ g'_y = 0 & \end{array}}$

② Degenerate constraint (NDCQ)



Ex: $x^2+y^2=25$

$g(x,y)=x^2+y^2$

$a=25$

Degenerate constraints.

exceptional cand. pts.

$$\left\{\begin{array}{l} g'_x=0 \\ g'_y=0 \\ g(x,y)=a \end{array}\right.$$

$2x=0 \Rightarrow x=0$

$2y=0 \Rightarrow y=0$

$x^2+y^2=25$

$0+0=25$

impossible.

$\Downarrow$

$\boxed{\text{no exceptional candidate pts}}$

Note: If an admissible pt has a tangent, then it does not have a degenerate constraint.

Ex: max $f(x,y)=y$ when $x^2+y^3=0$

Ordinary candidate pts:

$$\mathcal{L} = y - \lambda(x^2+y^3)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda\cdot 2x = 0 \\ \mathcal{L}'_y &= 1-\lambda\cdot 3y^2 = 0 \\ & x^2+y^3=0 \end{aligned}}$$

$\cancel{\lambda=0}$ or $\cancel{x=0}$

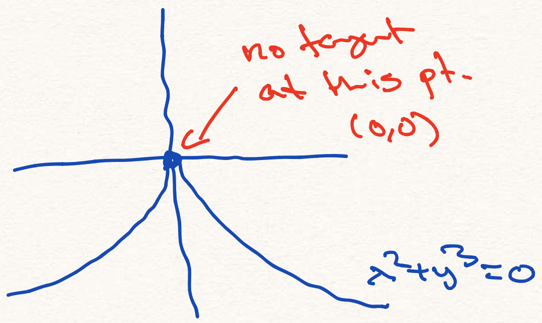
$1-0=0$ impossible

(3) $0+y^3=0$, $\underline{y=0}$

(2) $1-0=0$ impossible

no ordinary candidate pts

FOC + C $\Rightarrow$ no solutions.



$$x^2+y^3=0$$
$$y^3=-x^2$$
$$y=\sqrt[3]{-x^2}=-\sqrt[3]{x^2}$$

Exceptional cand. pts:

$g'_x = 2x = 0$ $\quad x=0$

$g'_y = 3y^2 = 0$ $\quad y=0$

$x^2+y^3=0$

$\Downarrow$

One pt. $\underline{(x,y)=(0,0)}$

$f_{max}=0$ at $(0,0)$

3 Global maxima / minima

Usual problems:

i) Difficult to solve FOC+C.

* $x^2 = k \Rightarrow x = \pm\sqrt{k}$ $(k>0)$
* if you divide by an expression that is not constant, you could lose solutions if the expr. can be zero.

$\Rightarrow$ factorize instead of dividing.

$2x\lambda = 0$ $\quad$ $2x \cdot \lambda = 0$

$x=0$ or $\lambda = 0$

ii) It is not sure that any of the candidate pts. are max/min.

EVT: D bounded $\Rightarrow$ there is max/min.

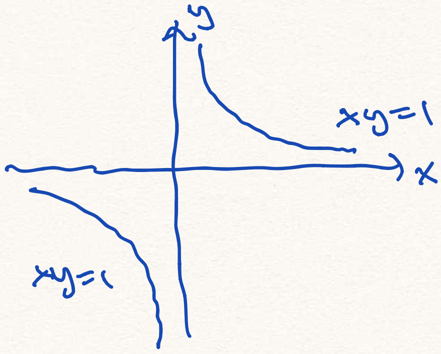
D is not bounded: there could be a max/min, but it is not sure.

iii) If there are exceptional candidate pts, they could be max/min.

Ex: max/min $f(x,y) = x^2+y^2$ when $xy=1$

D: $xy=1$

$y = 1/x$



Asymptotes:

$x=0$ (y-axis)

$y=0$ (x-axis)

D is not bounded

$\Downarrow$ cannot use EVT

not sure if there is a max/min.

Lagrange's method:

$L = x^2+y^2-\lambda xy$

$$\boxed{\begin{aligned} L'_x &= 2x-\lambda y = 0 \\ L'_y &= 2y-\lambda\cdot x = 0 \\ xy &= 1 \end{aligned}}$$

FOC + C

(1) $\frac{2x}{2} = \frac{\lambda y}{2}$

$x = \frac{\lambda y}{2}$

(2) $2y - \lambda\cdot\left(\frac{\lambda y}{2}\right) = 0 \quad \cdot 2$

$4y - \lambda^2 y = 0$

$y(4-\lambda^2) = 0$

$y(2-\lambda)(2+\lambda) = 0$

$y=0$ | $\lambda = 2$ | $\lambda = -2$

$$\begin{cases} 2x - \lambda y = 0 \\ 2y - \lambda x = 0 \\ xy = 1 \end{cases}$$

(1) $x = \dfrac{\lambda y}{2}$

(2) $y = 0$ or $\lambda = 2$ or $\lambda = -2$

$\cancel{y=0}$	$\lambda = 2$	$\cancel{\lambda = -2}$
$x = 0$	(1) $x = y$	(1) $x = -y$
(3) $0 \cdot 0 = 1$	(3) $x^2 = 1$	(3) $(-y)y = 1$
impossible	$x = \pm 1$	$y^2 = -1$
	$\Downarrow$	impossible
	$(1, 1; 2)$ $f = 2$	
	$(-1, -1; 2)$ $f = 2$	

Candidate pts:

i) Ordinary cand pts: $(1,1;2)$ $f=2$

$(-1,-1;2)$ $f=2$

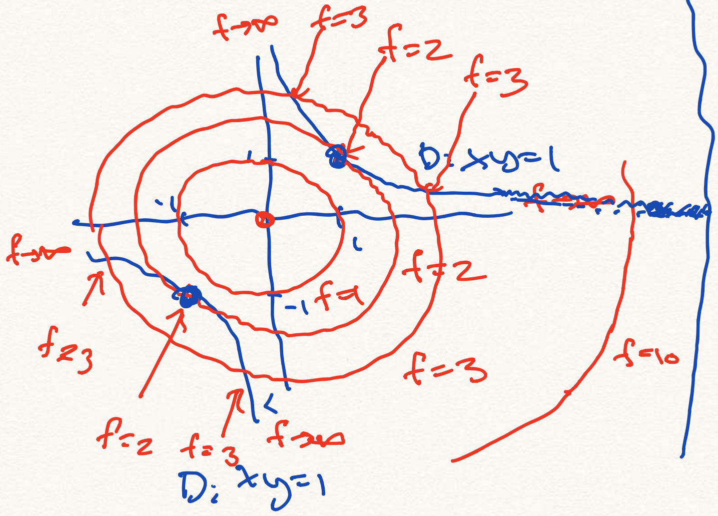
ii) Exceptional cand. pts: none (hyperbola)

$xy = 1$: $\left.\begin{array}{l} g'_x = y = 0 \\ g'_y = x = 0 \end{array}\right\} (x,y) = (0,0)$

$xy = 1$ $\quad 0 \cdot 0 = 1$ impossible

Use level curves: max/min $f=x^2+y^2$ when $xy=1$

Constraint: $xy=1$



Level curves for f:

$f(x,y)=c$

$x^2+y^2=c$

circles,
center $(0,0)$
radius $\sqrt{c}$ $(c>0)$

$c=0$: $(0,0)$

$c=1$: $x^2+y^2=1$

$c=2$: $x^2+y^2=2$

$c=3$: $x^2+y^2=3$

Conclusion:

$f_{min}=2$ at $(1,1)$, $(-1,-1)$

there is no max

($f\to\infty$ along the hyperbola)

$$\left.\begin{aligned} x&=a \\ y&=1/a \end{aligned}\right\}$$

$(a, 1/a)$ is admissible: $a\cdot\frac{1}{a}=1$

$f(x,y)=f(a,1/a)=a^2+1/a^2\to\infty$ as $a\to\infty$