Lecture 31 EBA 2910

Plan:

1. Lagrange problems
2. Lagrange multiplicators
3. Other optimization problems

Exam: June 19th

Review lecture: Around June 12th.

Info on exams: COVID-19 (lecture plan)

Review:

Lagrange problems: optimization (max/min) problems where all constraints are equations.

max/min $f(x,y)$ when $g(x,y)=a$

$f(x,y)$: obj. function

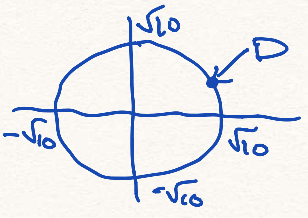
$g(x,y)=a$: equality constraint

Ex:

max/min $f(x,y) = x+3y$
when $x^2+y^2=10$

D: set of adm. pts
= all pts that satisfy all constraints

D: $x^2+y^2=10$



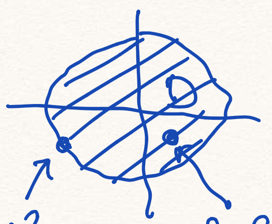
D is bounded

$-\sqrt{10} \le x \le \sqrt{10}$
$-\sqrt{10} \le y \le \sqrt{10}$

circle, center $(0,0)$, $r=\sqrt{10}$

D is closed ($=$), and all pts in D are boundary pts.

D: $x^2+y^2 \le 10$
(not Lagrange pb.)



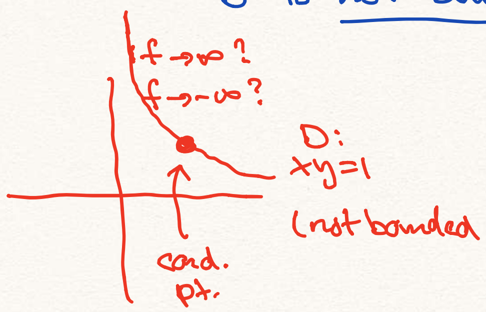
$x^2+y^2=10$ boundary pts.

$x^2+y^2<10$ interior pts.

① Methods to solve Lagrange problems:

D is bounded : EVT (extreme value thm.)
D: $g(x,y)=a$ $\Rightarrow$ there is a max/min.

D is not bounded : There may be a max/min, but it is not sure. If there is a max/min, it must be at one of the candidate pts.



② How to find candidate pts:

$$\mathcal{L} = f(x,y) - \lambda \cdot g(x,y)$$

i) Ordinary candidate pts: $(x,y;\lambda)$

Solve FOC + C (Lagrange cond.)

FOC: $\mathcal{L}'_x = 0$, $\mathcal{L}'_y = 0$

C: $g(x,y)=a$

ii) Exceptional candidate pts: (x,y)

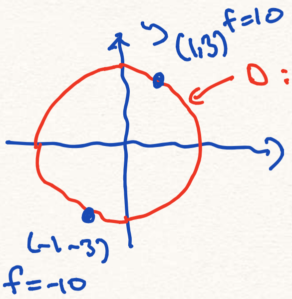
(adm. pts with degenerated constraint)

Solve :
$$\left.\begin{aligned} g'_x &= 0 \\ g'_y &= 0 \\ g(x,y) &= a \end{aligned}\right\} \quad \nabla g = \underline{0}$$

there is no proper tangent line

It is a good idea to compute f at each candidate pt as you find them.

Ex: max/min $f(x,y) = x + 3y$ wh $\underbrace{x^2+y^2}_{g(x,y)} = \underset{a}{10}$



$D: x^2+y^2=10$
circle, bounded $\Rightarrow$ there is a max/min
EVT

$D: \; x^2+y^2=10$

Candidate pts: $\quad L = x + 3y - \lambda \cdot (x^2+y^2)$

i) Ordinary:

$$\text{FOC}\begin{cases} L'_x = & 1 - \lambda \cdot 2x = 0 \\ L'_y = & 3 - \lambda \cdot 2y = 0 \\ \text{c} & x^2 + y^2 = 10 \end{cases}$$

(1) $\quad 1 = 2\lambda \cdot x \Rightarrow x = \dfrac{1}{2\lambda} \qquad (\lambda \neq 0)$

(2) $\quad 3 = 2\lambda \cdot y \Rightarrow y = \dfrac{3}{2\lambda}$

(3) $\quad \left(\dfrac{1}{2\lambda}\right)^2 + \left(\dfrac{3}{2\lambda}\right)^2 = 10$

$$\frac{1}{(2\lambda)^2} + \frac{9}{(2\lambda)^2} = 10 \qquad 1 \cdot (2\lambda)^2$$

$$10 = 10 \cdot (2\lambda)^2 \qquad 1 : 10$$

$$(2\lambda)^2 = 1 \Rightarrow 2\lambda = \pm\sqrt{1} = \pm 1$$

$$\lambda = \pm 1/2$$

Ord. cand. pts:

$\lambda = 1/2$:

$x = 1, \; y = 3$

$\lambda = 1/2, \; f = 10$

$(x,y;\lambda) = (1,3;1/2)$

$\lambda = -1/2$:

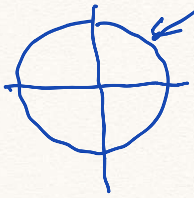
$x = -1, \; y = -3$

$\lambda = -1/2, \; f = -10$

$(x,y;\lambda) = (-1,-3;-1/2)$

ii) <u>Exceptional:</u> Degenerate constraint

$x^2+y^2=10$
(circle)



$$\left.\begin{aligned} g'_x &= 2x = 0 \\ g'_y &= 2y = 0 \end{aligned}\right\} \quad \nabla g = \underline{0}$$

$$\Rightarrow (x,y) = \underline{(0,0)}$$

$$x^2+y^2=10 \qquad 0^2+0^2 \neq 10$$

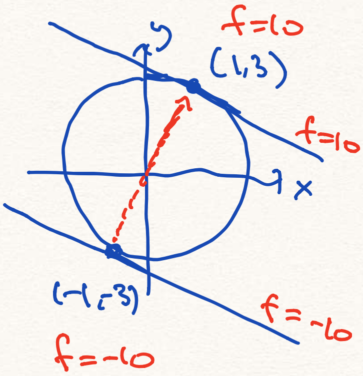
no except. cand. pts.

<u>Alt:</u> $x^2+y^2=10$ is a circle, and there is a proper tangent at each pt

$\Rightarrow$ <u>no except. cand. pts</u>

<u>Conclusion:</u> $f_{max} = \underline{\underline{10}}$ at $(x,y;\lambda) = (1,3;1/2)$

$f_{min} = \underline{\underline{-10}}$ at " $= (-1,-3;-1/2)$



since D (a circle) is bounded

<u>Level curves of f:</u>

$f(x,y)=c$
$x+3y=c$

$$3y = c - x$$
$$\underline{y = \frac{c}{3} - \frac{1}{3}x}$$

<u>c=10:</u> f=10
$y = \frac{10}{3} - \frac{1}{3}x$

<u>c=-10:</u> f=-10
$y = -\frac{10}{3} - \frac{1}{3}x$

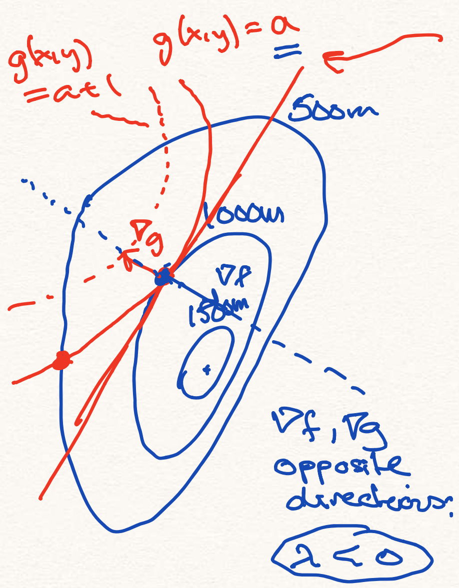
1 Lagrange problems:

$$\boxed{\begin{array}{l}\max/\min\ f(x,y)\\ \text{when}\quad g(x,y)=a\end{array}}$$

$h = f(x,y) - \lambda \cdot g(x,y)$

FOC: $\left\{\begin{array}{l} h'_x = 0 \\ h'_y = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} f'_x - \lambda \cdot g'_x = 0 \\ f'_y - \lambda \cdot g'_y = 0 \end{array}\right\}$ Why?

C: $\{\ g(x,y) = a$ ✓



tangent of the red curve (adm pts D: $g(x,y)=a$)

= tangent of the blue curve (level curve of $f(x,y)$)

Use the gradients to express this:

$$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \qquad \nabla g = \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

level curves for $f(x,y)$

set of adm pts

Note: the red and blue curve have the same tangent

$\Updownarrow$

$$\nabla f = \lambda \cdot \nabla g$$

$$\text{FOC} \boxed{\begin{array}{l} f'_x - \lambda g'_x = 0 \\ f'_y - \lambda \cdot g'_y = 0 \end{array}} \Leftrightarrow \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

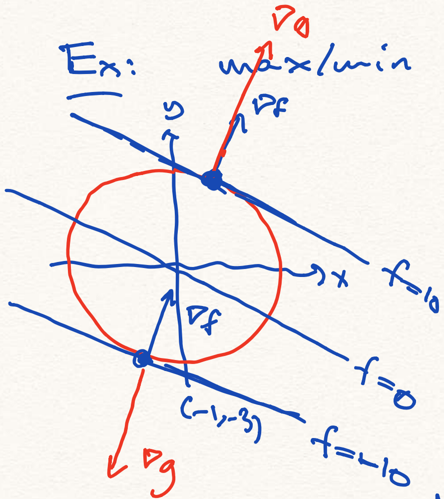
Comments:

i) The sign of λ has significance.

ii) The method of "same tangent line" and gradient breaks down at pts with $\nabla g = 0$.

(degenerate constraints)

Ex: max/min $f = x+3y$ when $\underbrace{x^2+y^2=10}_{\text{circle}}$



Level curves of f:

$f(x,y) = c$

$x+3y = c \Rightarrow \boxed{y = \frac{c}{3} - \frac{1}{3}x}$

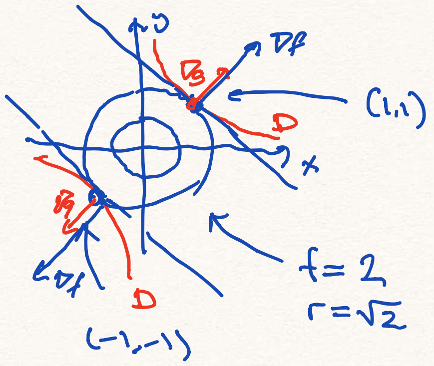
Cand. pts: $\boxed{\begin{aligned}(x,y;\lambda) &= (1,3;1/2) \quad f=10 \\ &= (-1,-3;-1/2) \quad f=-10\end{aligned}}$

$\nabla f = \begin{pmatrix}1\\3\end{pmatrix}$

$\nabla g = \begin{pmatrix}2x\\2y\end{pmatrix}$ $\left\{\begin{aligned}\nabla g(1,3) &= \begin{pmatrix}2\\6\end{pmatrix} \\ \nabla g(-1,-3) &= \begin{pmatrix}-2\\-6\end{pmatrix}\end{aligned}\right.$

Ex: max/min $f = x^2 + y^2$ when $xy = 1$

$D: y = 1/x$



Level curves of $f = x^2+y^2$:

$x^2+y^2 = c$ circle, center $(0,0)$, $r = \sqrt{c}$ $(c>0)$

$$\mathcal{L} = x^2+y^2 - \lambda xy$$

FOC $\begin{cases} \mathcal{L}'_x = 2x - \lambda y = 0 & (1) \\ \mathcal{L}'_y = 2y - \lambda x = 0 & (2) \end{cases}$

C $\{ \quad xy = 1$

(1) $2x = \lambda y \Rightarrow \underline{x = \frac{\lambda}{2} \cdot y}$

(2) $2y - \lambda \cdot \frac{\lambda}{2} y = 0 \quad | \cdot 2$

$4y - \lambda^2 y = 0$

$y(4-\lambda^2) = 0$

$y = 0$ or $\lambda^2 = 4$

no adm. pts with degenerate constraints since D is a <u>hyperbola</u>.

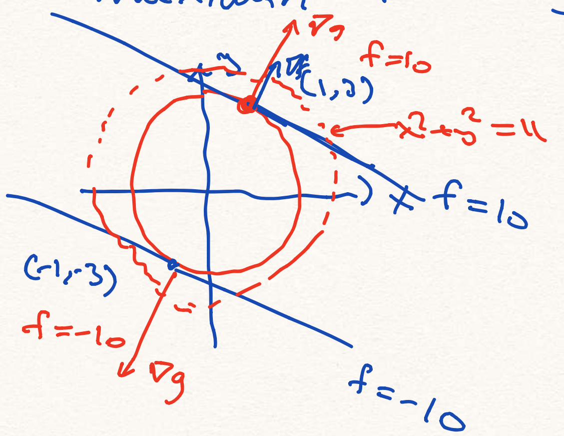
$y=0$	$\lambda = 2$	$\lambda = -2$
$x=0$	$x=y$	$x=-y$
$xy = 0\cdot 0 \neq 1$	$y^2 = 1$	$-y \cdot y = 1$
no cand. pts	$y = \pm 1$	$y^2 = -1$
	$(1,1,2)$ $f=2$	no cand. pts.
	$(-1,-1,2)$ $f=2$	
	ordinary cand. pts.	
	<u>min. pts</u>	

$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\nabla g = \begin{pmatrix} y \\ x \end{pmatrix}$

(2) Lagrange multipliers λ

max/min $f = x + 3y$ when $\underline{\underline{x^2 + y^2 = 10}}$.



Cand pts: (ordinary)

$(x, y; \lambda) = (1, 3; 1/2)$ $f = 10$

$(-1, -3; -1/2)$ $f = -10$

$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$: the direction in which g increases quickest.

$x^2 + y^2 = 11$: circle $r = \sqrt{11}$

What is the max/min value now?

Lagrange problem with parameter: max $f = x + 3y$ when $x^2 + y^2 = \underline{\underline{a}}$

Ex: $a = 10$ $\to$

$x^*(10) = 1$ $\lambda^*(10) = 1/2$

$y^*(10) = 3$ $\boxed{f^*(10) = 10}$

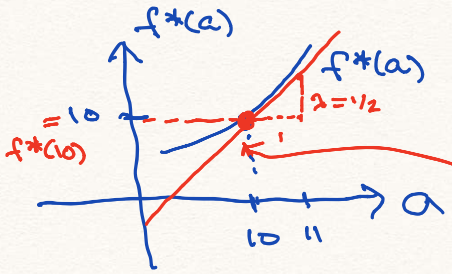
$a = 10$ f_{max} $(a = 10)$

Notation:

$(x^*(a), y^*(a); \lambda^*(a))$ — maximum candidate pt.

$f^*(a)$: maximum value

$f^*(a) = f(x^*(10), y^*(10))$



$a = 10$: $x^2 + y^2 = 10$

Interpretation of $\lambda = 1/2$

max $f = x + 3y$ when $x^2 + y^2 = 11$

is approximately

$10 + 1/2 = 10.5$

(when we increase a with 1, the maximal value increases with λ)

maximum value function: $f^*(a)$

Precise statement:

interpretation of λ

$$\boxed{\frac{df^*(a)}{da} = \lambda}$$

Slope of the tangent line is λ

$$f^*(11) \simeq f^*(10) + \Delta a \cdot \frac{df^*(a)}{da}$$

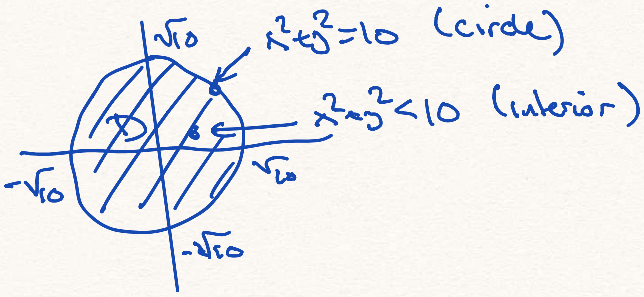
$f^*(10) = 10$, $\Delta a = (11 - 10) = 1$, $\frac{df^*(a)}{da} = \lambda = 1/2$

approximation since we follow the tangent (red) and not curve $f^*(a)$ (blue)

③ An example

max/min $f(x,y) = x + 3y$ when $x^2 + y^2 \le 10$

not Lagrange pb. (Kuhn–Tucker pb.)



D: $x^2+y^2 \le 10$

① D is closed ($\le$) and bounded

$$\left(\begin{array}{c} -\sqrt{10} \le x \le \sqrt{10} \\ -\sqrt{10} \le y \le \sqrt{10} \end{array} \right)$$

$\Downarrow$ EVT

there is max/min.

② Candidate pts:

i) Interior stationary pts:

$x^2+y^2<10$: $\left.\begin{array}{l} f'_x = 1 = 0 \\ f'_y = 3 = 0 \end{array}\right\}$ no stationary pts.

ii) Boundary pts:

$x^2+y^2=10$: Lagrange case

$(x,y;\lambda) = (1,3;1/2)$ $f=10$

$,(-1,-3;-1/2)$ $f=-10$

$\Downarrow$ D is compact

$f_{max} = \underline{\underline{10}}$ at $(1,3;1/2)$

$f_{min} = \underline{\underline{-10}}$ at $(-1,-3;-1/2)$