Lecture 32 EBA 2910

Plan:
1. General information
2. Information about the exam
3. Final exam 05/2019

1. Elective: ELE 3781 Mathematics

- mathematics
- python

2. Information about the exam:

How:
- one pdf, handwritten
- Wiseflow: message on It's L.
  email from BI
  - test
- think about how you write your solutions

Evaluation: grading the exam

40% - 60% to pass

What:
- many problems like the problems on previous exams
  - integration
  - matrices
  - fcn in several variables
  - derivation with appl. (max/min, tangent, convex/concave)
  - functions and graphs (lines, circle/ellipse, parabola, hyperbola, asymptotes)
  - present value comp. (discrete time / cont. time)

more pb's with these topics

19-20 problems

(*) There will be problems that are given geometrically, not by an algebraic expression.

3) Exam 2019/05:

1. $A = \begin{pmatrix} 1 & a & 4 \\ 2a & 8 & 12 \\ 5 & 10 & 16 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 11 \\ 40 \\ 51 \end{pmatrix}$

a) $A\underline{x} = \underline{b}$
$a = 2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 11 \\ 4 & 8 & 12 & 40 \\ 5 & 10 & 16 & 51 \end{array}\right) \begin{matrix} \\ \leftarrow -4 \\ \leftarrow -5 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 11 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right) \begin{matrix} \\ \\ \leftarrow -1 \end{matrix}$$

augmented matrix

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 11 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$x + 2y + 4z = 11$
$-4z = -4$

$z = 1$ $\quad x + 2y + 4 \cdot 1 = 11$
$x = 7 - 2y$

$$\boxed{\text{Solution: } (x, y, z) = (7-2y,\ y,\ 1) \text{ with } y \text{ free}}$$

b) $\begin{vmatrix} 1 & a & 4 \\ 2a & 8 & 12 \\ 5 & 10 & 16 \end{vmatrix} = 1 \cdot (8 \cdot 16 - 10 \cdot 12) - a(2a \cdot 16 - 5 \cdot 12) + 4(2a \cdot 10 - 8 \cdot 5)$

$= 128 - 120 - 32a^2 + 60a + 80a - 160$

cofactor expansion

$= -32a^2 + 140a - 152$

$|A| = 0$: $\quad -32a^2 + 140a - 152 = 0$

$$a = \frac{-140 \pm \sqrt{140^2 - 4 \cdot (-32)(-152)}}{2 \cdot (-32)}$$

$a = 2$ or $a = 19/8$

Break : Start again 13:10.

C) Find A^{-1} : $A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$

(a=3)

$$|A| = -32a^2 + 140a - 152$$

$$= -32 \cdot 3^2 + 140 \cdot 3 - 152 = \underline{-20}$$

$A = \begin{pmatrix} 1 & 3 & 4 \\ 6 & 8 & 12 \\ 5 & 10 & 16 \end{pmatrix}$:

$C_{11} = +(8 \cdot 16 - 10 \cdot 12) = 8$ $\quad C_{12} = -(6 \cdot 16 - 5 \cdot 12) = -36$ $\quad C_{13} = +(60 - 40) = 20$

$C_{21} = -8$ $\quad C_{22} = -4$ $\quad C_{23} = 5$

$C_{31} = -12$ $\quad C_{32} = 12$ $\quad C_{33} = -10$

$$A^{-1} = \frac{1}{-20} \cdot \begin{pmatrix} 8 & -36 & 20 \\ -8 & -4 & 5 \\ -12 & 12 & -10 \end{pmatrix}^T = \frac{1}{-20} \begin{pmatrix} 8 & -8 & -12 \\ -36 & -4 & 12 \\ 20 & 5 & -10 \end{pmatrix}$$

d) Show : $\boxed{A^7 \cdot \underline{x} = \underline{b} \text{ has exactly one solution for } \underline{a=-1}}$

Know from b) that $|A| \neq 0$ for $a = -1$.

$\Downarrow$

$A\underline{x} = \underline{b}$ has exactly one solution.

$A^7 \cdot \underline{x} = \underline{b}$

$|A^7| = |A \cdot A \cdot \ldots \cdot A| = |A|^7 \neq 0 \Rightarrow (A^7)^{-1}$ exists

$\cancel{(A^7)^{-1}} \cdot \cancel{A^7} \cdot \underline{x} = (A^7)^{-1} \cdot \underline{b}$

$\underline{x} = (A^7)^{-1} \cdot \underline{b}$

$= (A^{-1})^7 \cdot \underline{b}$

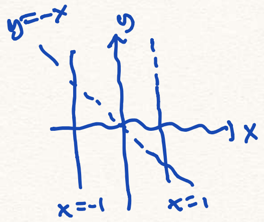
$= A^{-7} \cdot \underline{b}$ //

2. $f(x) = \dfrac{x^3}{1-x^2}$

a) Asymptotes of f: Vertical: $1-x^2=0$

$x=1$ or $x=-1$

$\boxed{\begin{array}{l} 1^3 = 1 \neq 0 \\ (-1)^3 = -1 \neq 0 \end{array}}$



Horizontal/Skew: x^3 : degree 3

$1-x^2$: " 2

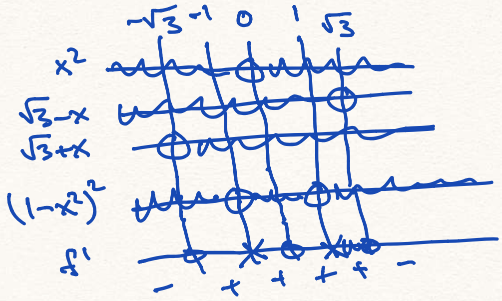
$$x^3 : (1-x^2) = x^3 : (-x^2+1) = -x$$
$$-(x^3-x)$$
$$x$$

$$\boxed{f(x) = -x + \frac{x}{1-x^2}} \to 0 \text{ when } x \to \pm\infty$$

Asymptote: $y = -x$

b) $$f'(x) = \left(\frac{x^3}{1-x^2}\right)' = \frac{3x^2\cdot(1-x^2) - x^3\cdot(-2x)}{(1-x^2)^2} = \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2}$$
$$= \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2} = \frac{x^2(\sqrt{3}-x)(\sqrt{3}+x)}{(1-x^2)^2}$$

f decreasing: $f' \leq 0$



f is decreasing in $[\sqrt{3}, \to)$ and in $(\leftarrow, -\sqrt{3}]$

3.

a) $\int 30(1-x)^5 dx = \int 30\, u^5 \frac{du}{-1}$ $\boxed{\begin{array}{c} u=1-x \\ du=-1 \cdot dx \end{array}}$

$\boxed{\text{power rule}} \longrightarrow$ $= -30 \; \frac{1}{6} u^6 + C = -5(1-x)^6 + C$

b) $\int \frac{12}{4-9x^2} dx = \int \frac{12}{(2+3x)(2-3x)} dx$

partial fractions

$\frac{12}{(2+3x)(2-3x)} = \frac{A}{2+3x} + \frac{B}{2-3x}$

$12 = A \cdot (2-3x) + B(2+3x)$

$12 = (2A+2B) + (3B-3A)x$

$\underset{12}{\overset{\shortparallel}{}}$ $\underset{0}{\overset{\shortparallel}{}}$

$3B-3A=0$ $A=B$

$2A+2B=12$ $4A=12$ $A=3, B=3$

$= \int \frac{3}{2+3x} + \frac{3}{2-3x} dx$

$= \ln|2+3x| - \ln|2-3x| + C$

$\boxed{\begin{array}{c} u=2+3x \\ du=3 \cdot dx \end{array}}$

$\int \frac{3}{2+3x} dx = \int \frac{\cancel{3}}{u} \cdot \frac{du}{\cancel{3}}$

$= \ln|u| + C$

c) $\int \frac{2e^x}{e^x - e^{-x}} dx = \int \frac{2\cancel{u}}{u - 1/u} \frac{du}{\cancel{u}}$

$\boxed{\begin{array}{c} u=e^x \\ du=e^x dx \end{array}}$

$= \int \frac{2 \quad \cdot u}{u - 1/u \quad \cdot u} du = \int \frac{2u}{u^2-1} du$

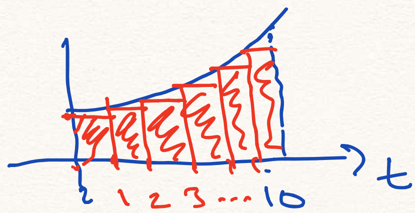
$$= \int \frac{2u}{u^2-1}\,du = \int \frac{\cancel{2u}}{v} \cdot \frac{dv}{\cancel{2u}}$$

$$\boxed{\begin{aligned} v &= u^2-1 \\ dv &= 2u \cdot du \end{aligned}}$$

$$= \int \frac{1}{v}\,dv = \ln|v| + C = \ln|u^2-1| + C$$

$$= \ln|e^{2x}-1| + C$$

4. Riemann sums



= red area

$= f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + \dots + f(9) \cdot 1$

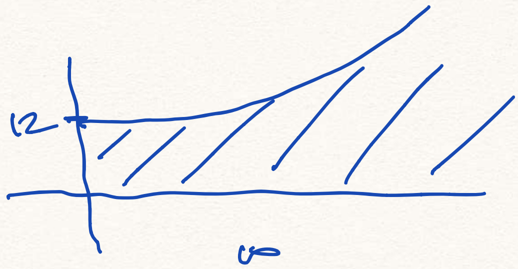
(rent paid once a year)

PV: $f(0) + \frac{f(1)}{(1+r)} + \frac{f(2)}{(1+r)^2} + \dots$

Continuous cash flow: rent paid "all the time"

$$PV = \int \frac{f(t)}{e^{rt}}\,dt$$

4. a) $I(t) = 12e^{0.07t}$



$$PV = \int_0^\infty \frac{I(t)}{e^{rt}}\,dt = \int_0^\infty \frac{12e^{0.07t}}{e^{0.10t}}\,dt$$

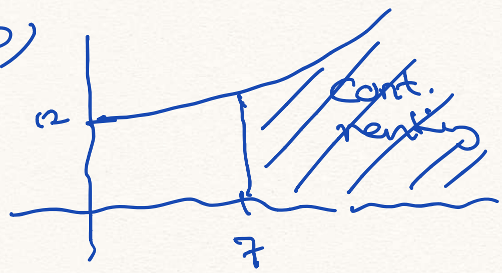
$$= \int_0^\infty 12 \cdot e^{-0.03t}\,dt \qquad \boxed{\begin{aligned} u &= -0.03t \\ du &= -0.03\,dt \end{aligned}}$$

$$= \left[12\left(\frac{1}{-0.03}\right)e^{-0.03t}\right]_0^\infty \qquad \boxed{\lim_{t\to\infty} e^{-0.03t} = 0}$$

$$= \frac{12}{-0.03}\cdot\left(0 - 1\right)$$

$$= \frac{12}{0.03} = \frac{12}{3/100} = \frac{1200}{3} = \underline{\underline{400}}$$

b)



Renting:

$$PV = \int_7^\infty 12e^{-0.03t}\,dt$$

$$= -400\left(0 - e^{-0.03\cdot 7}\right)$$

$$= 400e^{-0.21}$$

Sell: $S =$ Sale at $t=7$

$$PV = \frac{S}{e^{0.10\cdot 7}} = \frac{S}{e^{0.70}}$$

Compare: $\frac{S}{e^{0.70}} \geq 400 e^{-0.21}$

$$S \geq 400 \cdot e^{-0.21} \cdot e^{0.70}$$

$$= 400 \cdot e^{0.49} \simeq \underline{\underline{653}}$$

Break: Start 14.30.

5. $\boxed{f(x,y) = y^2 - x^3 + 3x}$

C: level curve of f thr. $(-1,2)$

$$f(-1,2) = 4 - (-1) + 3\cdot(-1) = 2$$

$$\boxed{C: \ f(x,y) = 2}$$

a) $\left.\begin{aligned} f'_x &= -3x^2 + 3 = 0 \\ f'_y &= 2y = 0 \end{aligned}\right\}$ FOC: $f'_x = f'_y = 0$

$\rightsquigarrow$ stationary pts.

$f'_x = -3x^2+3=0 \qquad x^2=1 \qquad x=\pm 1$

$f'_y = 2y=0 \qquad y=0$

Stat. pts: $\underline{\underline{(1,0)}}$, $\underline{\underline{(-1,0)}}$

Classify: Second derivative test.

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(f)(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det = -12 < 0 \Rightarrow$ (1,0) saddle pt.

$$H(f)(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$\left.\begin{array}{l} \det = 12 > 0 \\ tr = 8 > 0 \end{array}\right\} \Rightarrow$ (-1,0) local min

b) Tangent at C at (-1,2):

Slope: $y' = -\dfrac{f'_x(-1,2)}{f'_y(-1,2)} = -\dfrac{-3(-1)^2+3}{2\cdot 2} = \underline{0}$

Tangent: $y - y_0 = a\cdot(x-x_0)$

$y-2 = 0\cdot(x-(-1)) = 0$

$\underline{\underline{y=2}}$

Other intersection pts: $\quad y=2 \qquad C:\ f(x,y)=2$

$f(x,2)=2 \qquad -x^3+3x+2=0$

$2^2-x^3+3x=2 \qquad \boxed{x^3-3x-2=0}$ Eqn. of intersect. pts

$x^3 - 3x - 2 = 0$

$(x+1)\cdot(x^2 - x - 2) = 0$

know $x = -1$ is a solution

$\Downarrow$

$$\begin{array}{l} (x^3 - 3x - 2) : (x+1) = x^2 - x - 2 \\ -(x^3 + x^2) \\ \hline \quad -x^2 - 3x - 2 \\ \quad -(-x^2 - x) \\ \hline \qquad -2x - 2 \\ \qquad -2x - 2 \\ \hline \qquad\quad 0 \end{array}$$

$\underline{x = -1}$

or

$x^2 - x - 2 = 0$

$x = \dfrac{1 \pm \sqrt{1+8}}{2\cdot 1}$

$= \dfrac{1 \pm 3}{2}$

$x = 2$ or $x = -1$

<u>Intersection pts:</u>

$x = -1$, $x = -1$, $\underline{x = 2}$

Other intersection pts:

$x = 2 \quad y = 2$

$\Downarrow$

$(x, y) = \underline{\underline{(2, 2)}}$

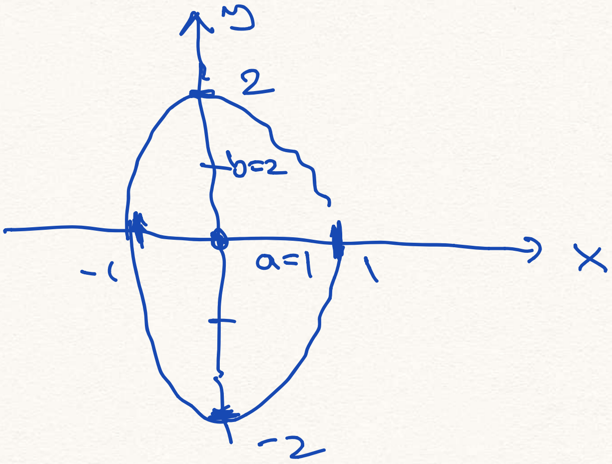
c) $4x^2 + y^2 = 4 \quad | : 4$

$$\frac{4x^2}{4} + \frac{y^2}{4} = \frac{4}{4}$$

$$x^2 + \frac{y^2}{4} = 1$$

$$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$$

<u>Ellipse</u>, center $(0,0)$, half-axis $a=1$, $b=2$



It is <u>bounded</u> since $-1 \le x \le 1$, $-2 \le y \le 2$ for all pts (x,y) on the ellipse

d)

max $f = y^2 - x^3 + 3x$

when $4x^2 + y^2 = 4$

i) Since this is an ellipse and <u>bounded</u>, the problem has a max by the extreme value thm.

ii) Candidate pts:

A) Ordinary: Lagrange's method

$$L = y^2 - x^3 + 3x - \lambda \cdot (4x^2 + y^2)$$

$$\text{FOC}\begin{cases} L'_x = -3x^2 + 3 - \lambda \cdot 8x = 0 \\ L'_y = 2y - \lambda \cdot 2y = 0 \end{cases}$$

$$\text{C}\left\{ \quad 4x^2 + y^2 = 4 \right.$$

ii) $2y - \lambda \cdot 2y = 0$

$2y(1-\lambda) = 0 \quad \rightarrow \quad \underline{y=0}$ or $\underline{\lambda = 1}$

$f(1/3, \pm\sqrt{32}/3) = \frac{32}{9} - \frac{1}{27} + 1$

$= \frac{96}{27} - \frac{1}{27} + \frac{27}{27} = \frac{122}{27} > 2$

$\underline{y=0}$:

iii) $4x^2 = 4$

$x^2 = 1 \quad x = \underline{\pm 1}$

i) $\cancel{-3 \cdot (\pm 3)} - \lambda \cdot 8 \cdot x = 0$

$-8\lambda \cdot x = 0$

$\lambda = 0 \quad (x \neq 0)$

$(x,y;\lambda) = \underline{(1,0;0)}, \ \underline{(-1,0;0)}$

$f = 2 \qquad f = -2$

$\underline{\lambda = 1}$: i) $-3x^2 - 8x + 3 = 0$

$x = \frac{8 \pm \sqrt{64 - 4 \cdot (-3) \cdot 3}}{2(-3)}$

$= \frac{8 \pm 10}{-6} = \underline{-3}, \underline{1/3}$

$\underline{x = -3}$: $4(-3)^2 + y^2 = 4$

$36 + y^2 = 4 \quad y^2 = -32$

not possible

$\underline{x = 1/3}$: $4 \cdot (1/3)^2 + y^2 = 4$

$y^2 = 4 - 4/9 = \frac{36}{9} - \frac{4}{9}$

$= 32/9$

$y = \pm\sqrt{\frac{32}{9}}$

$(x,y;\lambda) = (1/3, \pm\frac{\sqrt{32}}{3}; \underline{1})$ $f = \frac{122}{27}$

B) Exceptional candidate pts (degenerate constr.)

$\underbrace{4x^2+y^2=4}_{g(x,y)}$

$$\left.\begin{aligned} g'_x &= 8x = 0 \\ g'_y &= 2y = 0 \end{aligned}\right\} \begin{aligned} x&=0 \\ y&=0 \end{aligned}$$

$$\underbrace{4 \cdot 0^2 + 0^2}_{0} \neq 4$$

no such pts.

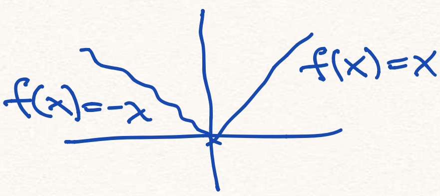
Conclusion: $f_{max} = f(1/3, \pm\sqrt{32}/3)$

$= 122/27$, $\lambda = 1$

Questions:

a) __Absolute value:__ $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

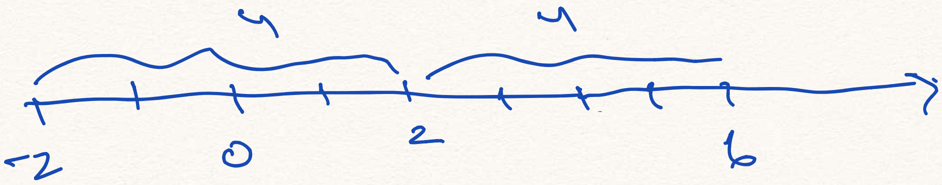
$|a|$: distance from a to 0



$y = |x|$

$|x-2| = 4 \iff x = 6$ or $x = -2$

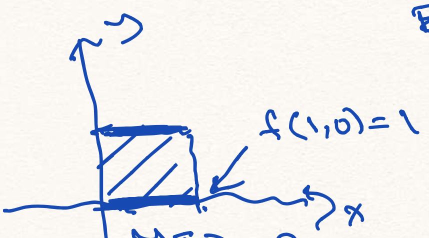
distance between x and 2 is 4



b) __Exam 2018/05__

4b) max $f(x,y) = (x-y)\,e^{2xy}$ wh $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$

EVT: there is a max



i) Boundary pt

ii) Interior stat pt.

i) $\left.\begin{array}{l} y=0 \\ 0 \leq x \leq 1 \end{array}\right\} \to f(x,0) = x \cdot e^0 = x$

ii) $f=(x-y)e^{2xy}$

$$\left.\begin{aligned} f'_x &= 0 \\ f'_y &= 0 \end{aligned}\right\} \quad \text{solve, check if interior pt} \left\{\begin{aligned} &0<x<1 \\ &0<y\le 1 \end{aligned}\right\}$$

i) Top edge:

$$\left.\begin{aligned} y&=1 \\ 0\le x&\le 1 \end{aligned}\right\} \quad f=(x-1)\cdot e^{2x\cdot 1}$$

$$=(x-1)e^{2x}$$

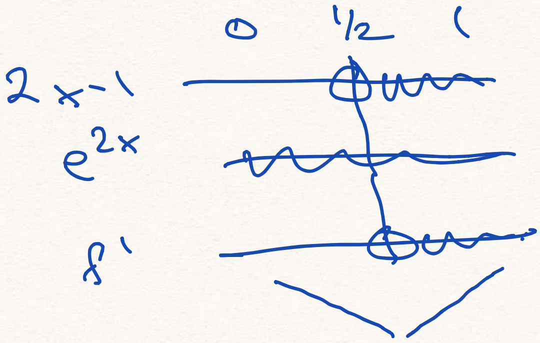
$$f'=1\cdot e^{2x}+(x-1)e^{2x}\cdot 2$$

$$=e^{2x}(1+2x-2)$$

$$=(2x-1)e^{2x}$$

$f(0,1)=-1$

$f(1,1)=0$



On this edge,

$f(1,1)=0$ is the max value.

6. $\max \quad f = p \ln(1+ax) + q \cdot \ln(1-bx)$

p, q, a, b const.

$$f'(x) = p \cdot \frac{1}{1+ax} \cdot a + q \cdot \frac{1}{1-bx} (-b)$$

$$= \frac{ap}{1+ax} + \frac{-bq}{1+bx}$$

$$= \frac{ap(1-bx)}{(1+ax)(1-bx)} + \frac{-bq(1+ax)}{(1-bx)(1+ax)}$$

$q = 1-p$

$ap - bq > 0$

$a, b > 0$

$$= \frac{ap - apbx - bq - bqax}{(1+ax)(1-bx)}$$

$$= \frac{\overbrace{ap-bq}^{>0} - ab(\overset{1}{p+q})x}{(1+ax)(1-bx)}$$

$f'(x) = 0: \quad ap - bq = abx$

$$x^* = \frac{ap-bq}{ab}$$

Exam 05/2019, Pb. 6:

$$\min f(x,y) = x \quad \text{u.b.c.} \quad \overbrace{y^2 - x^3 + 3x = 2}^{C}$$

Lagrange: $\mathcal{L} = x - \lambda(y^2 - x^3 + 3x)$

$$\boxed{\begin{aligned} \mathcal{L}'_x = \ & 1 - \lambda(-3x^2+3) = 0 \\ \mathcal{L}'_y = \ & -\lambda \cdot 2y = 0 \\ & y^2 - x^3 + 3x = 2 \end{aligned}}$$

ii) $-\lambda \cdot 2y = 0$

$\lambda = 0$ or $y = 0$

i) $1 - 0(\ldots) = 0$

$1 = 0$

impossible

$y = 0$: iii) $-x^3 + 3x = 2$

$x^3 - 3x + 2 = 0$

Try: $x = 1$ ok

$(x^3 - 3x + 2) : (x-1) = x^2 + x - 2$

$(x-1)(x^2+x-2) = 0$

$x = 1$ or $x^2 + x - 2 = 0$

$x = \dfrac{-1 \pm \sqrt{1+8}}{2}$

$x = 1$ or $x = -2$

~~$x = 1, y = 0$:~~

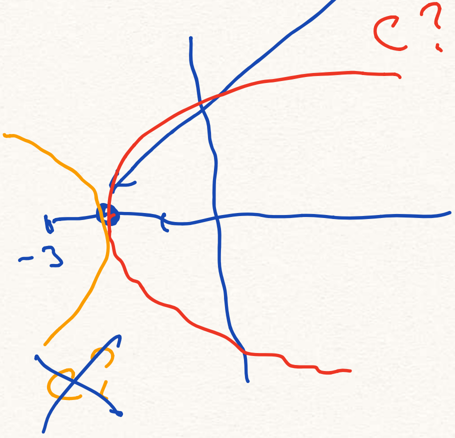
~~$1 - \lambda \cdot 0 = 0$~~

$x = -2; y = 0$:

$1 - \lambda(-9) = 0 \quad \lambda = -1/9$

$\Rightarrow$ <u>ord. cond. pt:</u> $(x, y; \lambda) = (-2, 0; -1/a)$

$\underline{f = -2}$

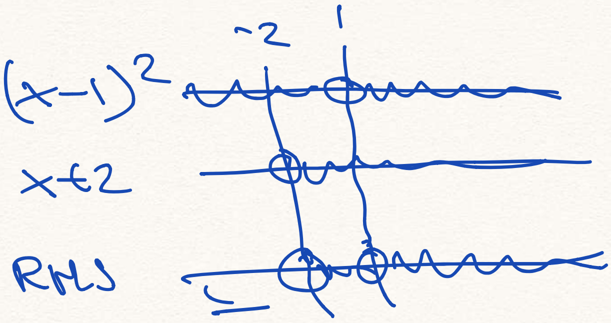


Try to show that $f = -2$ is the min.

$C: \quad y^2 - x^3 + 3x = 2$

$y^2 = x^3 - 3x + 2$

$y^2 = (x-1)(x-1)(x+2)$



$x < -2$: $y^2 =$ negative no. impossible.

<u>Concl:</u> $f_{min} = \underline{-2}$ at $(-2, 0, -1/a)$