EBA 2911, lecture 4, 4 Sept 2019, Runar Ile

Plan
1. Repetition and problems from last week
2. Infinite series and limit values
3. Euler's number and continuous compounding

1. Repetition | problems

Interest formula:
$$K_n = K_0 \cdot \overbrace{\underbrace{(1+r)}_{\substack{\text{growth factor} \\ \text{for 1 period}}}^{n}}^{\substack{\text{growth factor} \\ \text{for } n \text{ periods}}}$$

K_0 = deposit

K_n = balance after n interest periods

r = period rate

Prob 2b: Deposit: 50 000

Nominal interest: 3.6%

monthly compounding

i) Balance after 10 years:

$$50\,000 \cdot \left(1 + \frac{3.6\%}{12}\right)^{120} = 50\,000 \cdot 1.003^{120}$$
$$= \underline{\underline{71\,627.86}}$$

ii) Growth factor for 10 years:

$$1.003^{120} = \underline{\underline{1.4326}}$$

Relative change (10 years): $1.4326 - 1 = \underline{\underline{43.26\%}}$

iii) Annual growth factor: $1.003^{12} = 1.0366$

Effective annual interest: $1.0366 - 1 = \underline{\underline{3.66\%}}$

Cash flow : Present value and internal rate of return

Prob. 5

year	0	4	7
payment	-20	9	14

– a cash flow

a) Discount rate : 12 %. Then the present value of the cash flow is

$$-20 + \frac{9}{1.12^4} + \frac{14}{1.12^7} = \underline{\underline{-7.95}}.$$

b) A lower discount rate r will make the present values $\frac{9}{(1+r)^4}$ and $\frac{14}{(1+r)^7}$ <u>bigger</u>.

Hence the discount rate must be lowered for the present value of the cash flow to become 0.

<u>Alternative</u>: Since the present value is negative the revenue of the investment is less than 12 %.

c) We put $r = 2.44\%$ into the expression for the total present value of the cash flow and get:

$$-20 + \frac{9}{1.0244^4} + \frac{14}{1.0244^7} = 0.00$$

Then 2.44 % is the internal rate of return of the cash flow.

d) 2.44 % is far from 12 %. The investment is probably not so interesting. If the payment of 20 mill. is changed to 12.05 mill. then the investment would have 12 % internal rate of return.

Regular cash flows:

A fixed amount is paid each period.

Ex: Annuity loan (present value: what you can borrow)

Ex: Saving with a fixed amount each period. Future value: The balance (what you have saved)

The present value and the future value of a regular cash flow are geometric series.

Ex (annuity loan, 4%)

0	1	2	...	14	15
	70	70	...	70	70

$$\frac{70}{1.04}$$

$$\frac{70}{1.04^2}$$

$$\vdots$$

$$\frac{70}{1.04^{14}}$$

$$\frac{70}{1.04^{15}}$$

The sum is the total present val. of the regular cash flow which gives what ~~the bank~~ you can borrow.

Geometric series:

$$\frac{70}{1.04} + \frac{70}{1.04^2} + \dots + \frac{70}{1.04^{14}} + \frac{70}{1.04^{15}}$$

We read the series backwards:

$$a_1 = \frac{70}{1.04^{15}}, \quad k = 1.04, \quad n = 15$$

Then the present value (the sum) is given by the formula

$$a_1 \cdot \frac{k^n - 1}{k - 1} = \frac{70}{1.04^{15}} \cdot \frac{1.04^{15} - 1}{1.04 - 1} = \underline{\underline{778.29}}$$

2. Infinite series and limit values

<u>EX</u>: The annuity: 70 000, interest: 4 %

Number of years: n

Total present value:

$$\frac{70}{1.04^n} \cdot \frac{1.04^n - 1}{0.04} = \frac{70 \cdot (1.04^n - 1)}{1.04^n \cdot 0.04}$$

$$= \frac{70 \cdot (1.04^n - 1) \quad : 1.04^n}{\cancel{1.04^n} \cdot 0.04 \quad : \cancel{1.04^n}} = \frac{70 \cdot \left(\frac{1.04^n}{1.04^n} - \frac{1}{1.04^n}\right)}{0.04}$$

$$= \frac{70\left(1 - \frac{1}{1.04^n}\right)}{0.04} \xrightarrow[n\to\infty]{} \frac{70\cdot(1-0)}{0.04}$$

approaches 0 when $n \to \infty$

$$= \frac{70}{0.04} = \underline{\underline{1750}}$$

Conclusion: If you pay the bank 70000 each year forever, starting next year, with 4% interest, you can borrow 1.75 mill.

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3. Euler's number and continuous compounding

Ex: You deposit 1000 into an account with 12% nominal interest.

compounding	balance
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quarterly	$1000 \cdot 1.03^4 = 1125.51$
Monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1 + \frac{0.12}{365}\right)^{365} = 1127.47$
Pattern (n periods)	$1000 \cdot \left(1 + \frac{0.12}{n}\right)^n$

Euler's number: $e = 2.71828....$

calculate: $1000 \cdot e^{0.12} = \underline{1127.50}$

$1000 \boxed{\times}\ 0.12 \boxed{e^x} \boxed{=}$

Euler's number is defined as the limit of $\left(1+\frac{1}{n}\right)^n$ when n approaches ∞ ("becomes bigger and bigger")

Write: $$\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$$

<u>Ex</u>: $$\left(1+\frac{1}{1000}\right)^{1000} = 2.71692\ldots$$

$$\left(1+\frac{1}{1\text{ mill}}\right)^{1\text{ mill}} = 2.718280\ldots$$

Back to ex with $r = 12\%$:

$$\left(1+\frac{0.12}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^n$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\text{approaches } e \text{ when } n\to\infty}\Bigg]^{0.12}$$

So $$\left(1+\frac{0.12}{n}\right)^n \xrightarrow[n\to\infty]{} e^{0.12}$$

After 1 year with 12% nominal interest and <u>continuous compounding</u> the deposit of 1000 has increased to

$$1000 \cdot e^{0.12} = 1127.50$$

the annual growth factor!

The annual growth factor (with continuous compounding) is

$$e^{0.12} = 1.127497$$

The effective annual interest is

$$e^{0.12} - 1 = 0.127497 = 12.7497\,\%$$

After 2 years:

$$1000 \cdot e^{0.12} \cdot e^{0.12} = 1000 \cdot (e^{0.12})^2$$

$$= 1000 \cdot e^{0.12 \cdot 2}$$

$$= 1000 \cdot e^{0.24}$$

$$= \underline{\underline{1271.25}}$$

Problem: You deposit 10 mill. into an account with nominal interest: 2.8 %
calculate the balance after 5 years with

a) Annual compounding
b) Continuous compounding
c) Compute the effective annual interest with continuous compounding.

Solution:

a) Annual growth factor: 1.028

Balance after 5 years: $10 \text{ mill} \cdot 1.028^5$

$= \underline{\underline{11.48 \text{ mill}}}$

b) Annual growth factor: $e^{0.028} = 1.0284$

Balance after 5 years: $10 \text{ mill} \cdot (e^{0.028})^5$

$= 10 \text{ mill} \cdot e^{0.028 \cdot 5}$

$= 10 \text{ mill} \cdot e^{0.140}$

$= \underline{\underline{11.50 \text{ mill}}}.$

c) The effective interest is $e^{0.028} - 1$

$= 1.0284 - 1 = \underline{\underline{2.84\,\%}}.$