EBA 2911, lecture 5, 11 Sept. 2019, Runar Ile

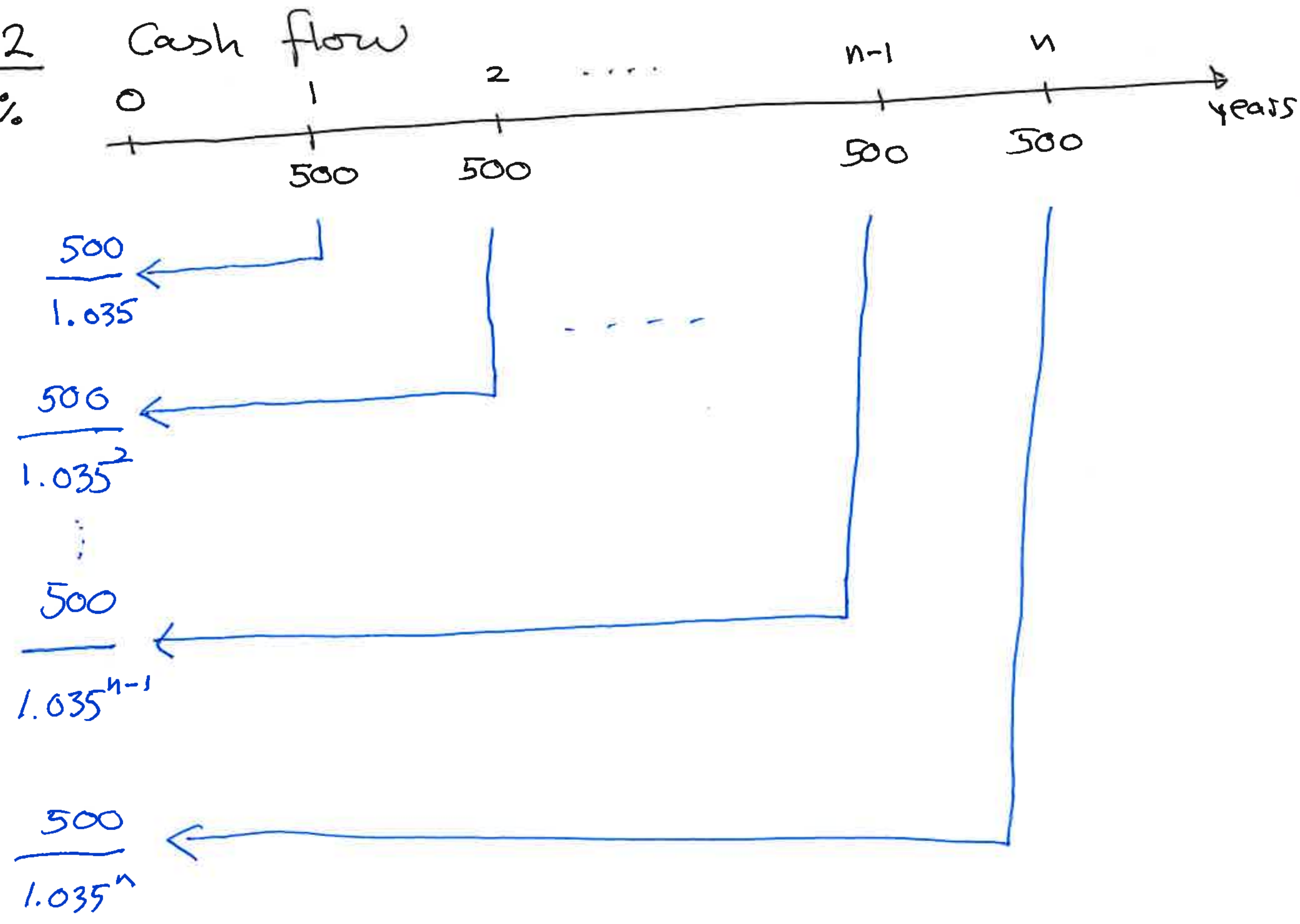
Plan:
1. Repetition & exercises
2. Linear & quadratic equations
3. Equations with parameters

1. Rep. & exercises

Financial mathematics $\begin{cases} \text{growth factor} \\ \text{present value} \\ \text{cash flow} \end{cases}$

Prob. 2 Cash flow

$r = 3.5\%$



Sum = total present value of cash flow.

a) Present value: $\frac{500}{1.035} + \frac{500}{1.035^2} + \dots + \frac{500}{1.035^n}$

This is a geometric series in two ways.

left to right: $a_1 = \frac{500}{1.035}$, $k = \frac{1}{1.035}$, $n = \#$ terms

then the sum is $a_1 \cdot \frac{k^n - 1}{k-1} = \frac{500}{1.035} \cdot \frac{\left(\frac{1}{1.035}\right)^n - 1}{\left(\frac{1}{1.035} - 1\right)}$

right to left: $a_1 = \frac{500}{1.035^n}$, $k = 1.035$, $n = \#$ terms

the sum: $a_1 \cdot \frac{k^n - 1}{k - 1} = \frac{500}{1.035^n} \cdot \frac{1.035^n - 1}{0.035}$

b) Insert the different values of n:

years n	10	20	40	80	1000
pres. value	4,158,302	7,106,201	10,677,536	13,374,387	14,285,714.29

c) The present value of never-ending payments is the number that the sum of n terms approaches more and more as n grows:

$$\frac{500'}{1.035^n} \cdot \frac{1.035^n - 1}{0.035} = \frac{500' \cdot (1.035^n - 1) : 1.035^n}{1.035^n \cdot 0.035 : 1.035^n}$$

$$= \frac{500' \cdot \left(1 - \frac{1}{1.035^n}\right)}{0.035} \xrightarrow[n \to \infty]{} \frac{500' \cdot (1 - 0)}{0.035}$$

$$= \frac{500'}{0.035} = 14,285,714.29$$

(since $1.035^n \xrightarrow[n \to \infty]{} \infty$, $\frac{1}{1.035^n} \xrightarrow[n \to \infty]{} 0$)

Prob. 6 a) 30 mill 5 years from now
annual interest : 13%
continuous compounding.

Annual growth factor : $e^{0.13} = 1.1388$

Present value : $\dfrac{30 \text{ mill}}{(e^{0.13})^5} = \dfrac{30 \text{ mill}}{e^{0.13 \cdot 5}} = \dfrac{30 \text{ mill}}{e^{0.65}}$

$= \underline{\underline{15.66 \text{ mill}}}$.

b) Cash flow:

Year	0	1	5	6	7
Paym.	-70	-20	30	55	80

The present value of the cash flow is the sum of the pres. values of each payment :

$$-70 - \frac{20}{e^{0.13}} + \frac{30}{e^{0.13 \cdot 5}} + \frac{55}{e^{0.13 \cdot 6}} + \frac{80}{e^{0.13 \cdot 7}} = \underline{\underline{-14.49}}$$

$\dfrac{20}{e^{0.13}} = 20 \cdot e^{-0.13}$

c) One will not earn 13% interest (with cont. comp.) on this investment

d) IRR (internal rate of return) is approx. 10% (w. cont. comp.) because the present value

$$-70 - \frac{20}{e^{0.1}} + \frac{30}{e^{0.1 \cdot 5}} + \frac{55}{e^{0.1 \cdot 6}} + \frac{80}{e^{0.1 \cdot 7}}$$ is approx. zero.

e) If the first payment in (b) was $70 - 14.49 = 55.51$ then the invest. would have IRR $= 13\%$ (cont. comp.)

f) Future value after 7 years (b):

$$-70 \cdot e^{0.13 \cdot 7} - 20 \cdot e^{0.13 \cdot 6} + 30 \cdot e^{0.13 \cdot 2} + 55 \cdot e^{0.13} + 80$$

$$= \underline{\underline{-35.99}}.$$

$$= e^{0.13 \cdot 7} \underbrace{\left(-70 - \frac{20}{e^{0.13}} + \frac{30}{e^{0.13 \cdot 5}} + \frac{55}{e^{0.13 \cdot 6}} + \frac{80}{e^{0.13 \cdot 7}} \right)}_{\text{present value}}$$

$$= e^{0.13 \cdot 7} \cdot (-14.49)$$

In the case (e) with IRR $= 13\%$ (cont. comp) the present value $= 0$ and then the future value $= e^{0.13 \cdot 7} \cdot 0 = \underline{\underline{0}}$

2. Linear and quadratic equations

A linear expression: $ax + b$ (a and b are numbers)

Ex: $4x - 3$ $(a = 4, \; b = -3)$.

A linear equation: An equation which can be transformed to an equation:

$$ax + b = 0 .$$

Ex: $\dfrac{1}{x+3} = \dfrac{2}{x+4}$ | $\cdot (x+3)(x+4)$ Multiply with a common factor on both sides

$x + 4 = 2(x+3)$ (distr. law)

$x + 4 = 2x + 6$ subtract $2x+6$ on b.s.

$-x - 2 = 0$ (a linear eq. with $a = -1$, $b = -2$)

$(x \neq -3 \text{ and } x \neq -4)$

A quadratic expression: $ax^2 + bx + c$ (a, b, c are numbers) parameters

A quad. equation: An eq. which can be transformed to the standard form: $ax^2 + bx + c = 0$

Ex: $3x + 9 = (x-1)(x+3)$

resolve the parantheses so that we can collect terms

$$3x + 9 = x^2 + 3x - x - 3$$

subtract $3x+9$ on both sides

$$x^2 - x - 12 = 0 \qquad (a=1,\ b=-1,\ c=-12)$$

Ex: $\frac{1}{x} + \frac{2}{x+1} = 3 \quad \Big| \cdot x(x+1)$

$$x+1 + 2x = 3 \cdot x(x+1)$$

$$3x + 1 = 3x^2 + 3x$$

subtract $3x^2 + 3x$ on b.s.

$$-3x^2 + 1 = 0 \qquad (a=-3,\ b=0,\ c=1)$$

$(x \neq 0$ and $x \neq -1)$

3. Equations with parameters

If $a \neq 0$ the quadratic formula (abc-formula) gives the solutions to the (any) quadratic equation on std. form: $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex: $3x^2+4x-5=0$ $(a=3,\ b=4,\ c=-5)$

the quadratic formula gives

$$x = \frac{-4 \pm \sqrt{4^2-4\cdot 3\cdot(-5)}}{2\cdot 3}$$

$$= \frac{-4 \pm \sqrt{16+60}}{6} = \frac{-4\pm\sqrt{76}}{6}$$

$$= \frac{-4 \pm \sqrt{4\cdot 19}}{6} = -\frac{4}{6} \pm \frac{2\sqrt{19}}{6}$$

$$= -\frac{2}{3} \pm \frac{\sqrt{19}}{3}.$$

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three cases:

$b^2-4ac>0$ gives two solutions

$b^2-4ac=0$ gives one solution

$b^2-4ac<0$ no solutions

Problem: Determine the number of solutions of the equations:

a) $x^2+5x+4.6=0$ — $5^2-4\cdot1\cdot4.6>0$: two solut's.

b) $-x^2+2x-1=0$ — $2^2-4\cdot(-1)(-1)=0$: one solut.

c) $4x^2-5x-5=0$ — $(-5)^2-4\cdot4\cdot(-5)>0$: two solut.

The quadratic formula is often inefficient

Ex: $-3x^2 + 7 = 0$ $\quad (a=-3,\ b=0,\ c=7)$

subtr. 7 from b. s.

$$-3x^2 = -7$$

divide by -3 on b. s.

$$x^2 = \frac{-7}{-3} = \frac{7}{3}$$

$$|x| = \sqrt{x^2} = \sqrt{\frac{7}{3}}$$

so $\quad x = \pm\sqrt{\frac{7}{3}}$

Ex: $2x^2 - 6x = 0$ $\quad (a=2,\ b=-6,\ c=0)$

$$2(x^2 - 3x) = 0$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

either $x = 0$ or $x - 3 = 0$

$$x = 3$$

$\boxed{a \cdot b = 0 \text{ implies that } a=0 \text{ or } b=0 \text{ (or both)}}$

Completing the square

Ex: $x^2 + 6x - 16 = 0$

Claim: $x^2 + 6x = (x+3)^2 - 9$

-because $(x+3)^2 = x^2 + 6x + 9$

Then the equation can be written as

$$(x+3)^2 - 9 - 16 = 0$$

$$(x+3)^2 = 9 + 16 = 25$$

so $x + 3 = 5$ or $x + 3 = -5$

$x = \underline{\underline{2}}$ or $x = \underline{\underline{-8}}$

If r_1 and r_2 er solutions ('roots') to the quadratic equation $x^2 + bx + c = 0$

then $(x - r_1)(x - r_2) = x^2 - r_2 x - r_1 x + r_1 r_2$

$$= x^2 - (r_1 + r_2)x + r_1 r_2$$

In ex. above: $r_1 + r_2 = -6$ and $r_1 r_2 = -16$

$2 + (-8) = -6$ and $2 \cdot (-8) = -16$