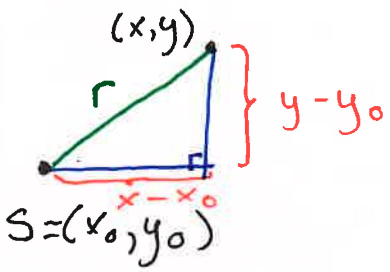
EBA 2911, lecture 9, 4 Oct. 2019, Runar Ile

1. Repetition & Problems
2. Rational functions, hyperbolas
3. Continuity and the intermediate value theorem
4. Composing functions

1. Rep. & Prob.

Problem 1c) $S = (-3.5, -3)$, $r = 2.5$

Pattern: $(x-x_0)^2 + (y-y_0)^2 = r^2$



$$(x-(-3.5))^2 + (y-(-3))^2 = 2.5^2$$

$$\underline{\underline{(x+3.5)^2 + (y+3)^2 = 6.25}}$$

2f) $25x^2 + 25y^2 - 20x - 30y = -12$

collect x-es and y-s

$$25\left(x^2 - \frac{20}{25}x\right) + 25\left(y^2 - \frac{30}{25}y\right) = -12$$

complete the squares inside the parantheses

$$25\left(\left(x-\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2\right) + 25\left(\left(y-\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2\right) = -12$$

$$25\left(x-\frac{2}{5}\right)^2 + 25\left(y-\frac{3}{5}\right)^2 = -12 + 25\cdot\frac{4}{25} + 25\cdot\frac{9}{25}$$

divide by 25 on each side $\qquad = -12+4+9 = 1$

$$\left(x-\frac{2}{5}\right)^2 + \left(y-\frac{3}{5}\right)^2 = \frac{1}{25}$$

So the centre $\underline{\underline{S = \left(\frac{2}{5}, \frac{3}{5}\right), \text{ radius } r = \sqrt{\frac{1}{25}} = \frac{1}{5}}}$

3b) Ellipses :

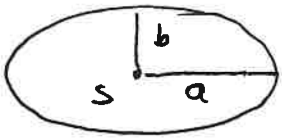
Pattern : $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

$S = (x_0, y_0)$, a and b are the semi-axes

Read off : $S = (15, 10)$, $a = 15$ and $b = 10$

$$\frac{(x-15)^2}{15^2} + \frac{(y-10)^2}{10^2} = 1 \quad \text{or}$$

$$\frac{(x-15)^2}{225} + \frac{(y-10)^2}{100} = 1$$



4g) $25x^2 + 4y^2 - 100x - 40y = -100$

collect x-es and y-s

$$25\left(x^2 - \overset{=4}{\frac{100}{25}}x\right) + 4\left(y^2 - \overset{=10}{\frac{40}{4}}y\right) = -100$$

complete the squares inside the paranthesss

$$25\left((x-2)^2 - 2^2\right) + 4\left((y-5)^2 - 5^2\right) = -100$$

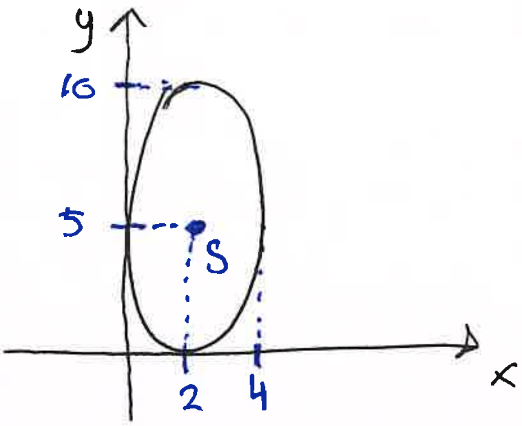
'move' the constants to the right h.s.

$$25(x-2)^2 + 4(y-5)^2 = -100 + 25 \cdot 4 + 4 \cdot 25 = 100$$

divide by 100 on each side

$$\frac{(x-2)^2}{4} + \frac{(y-5)^2}{25} = 1$$

So $S = (2, 5)$, $a = \sqrt{4} = 2$ and $b = \sqrt{25} = 5$



2. Rational functions, hyperbolas & asymptotes

Rational function: $f(x) = \dfrac{p(x)}{q(x)}$ polynomials

Ex: $f(x) = \dfrac{2x+1}{x^2+3}$

Because $f(x) = \dfrac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \dfrac{\frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}} \xrightarrow[x \to \infty]{} \dfrac{0^+}{1^+} = 0^+$

divide by x^2 in numerator and denominator

$\xrightarrow[x \to -\infty]{} \dfrac{0^-}{1^+} = 0^-$

This means that the line $y = 0$ (= the x-axis) is a horizontal asymptote for $f(x)$.

Ex: $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$ (Note: $f(x)$ is not def'd for $x = 1$, $x = 5$

If $x \to 1^-$ (x approaches 1 from below)

then $(x-1) \longrightarrow 0^-$

and $2x+1 \longrightarrow 2 \cdot 1 + 1 = 3$

and $x-5 \longrightarrow 1-5 = -4$ so

$$f(x) = \frac{\overbrace{2x+1}^{\to 3}}{\underbrace{(x-1)}_{0^-}\underbrace{(x-5)}_{-4}} \xrightarrow[x \to 1^-]{} +\infty$$

If $x \to 1^+$ (x approaches 1 from above)

then $(x-1) \longrightarrow 0^+$

$(2x+1) \longrightarrow 3$

$(x-5) \longrightarrow -4$

Then $$f(x) = \frac{\overbrace{2x+1}^{\to 3}}{\underbrace{(x-1)}_{0^+}\underbrace{(x-5)}_{-4}} \xrightarrow[x \to 1^+]{} -\infty$$

Problem: Investigate what happens with $f(x)$ when $x \to 5^-$ and $x \to 5^+$.

Solution: $2x+1 \xrightarrow[x \to 5]{} 2 \cdot 5 + 1 = 11$

$x-1 \xrightarrow[x \to 5]{} 5-1 = 4$

$$x-5 \xrightarrow[x\to 5^-]{} 0^- \quad \text{so} \quad f(x) = \frac{\overbrace{2x+1}^{\to 11}}{\underbrace{(x-1)}_{4}\underbrace{(x-5)}_{0^-}} \xrightarrow[x\to 5^-]{} -\infty$$

$$x-5 \xrightarrow[x\to 5^+]{} 0^+ \quad \text{so} \quad f(x) = \frac{\overbrace{2x+1}^{\to 11}}{\underbrace{(x-1)}_{4}\underbrace{(x-5)}_{0^+}} \xrightarrow[x\to 5^+]{} +\infty$$

Ex: $f(5.01) = 274.81$

$f(4.99) = -275.19$

$f(5.001) = 2749.81$, $f(4.999) = -2750.19$

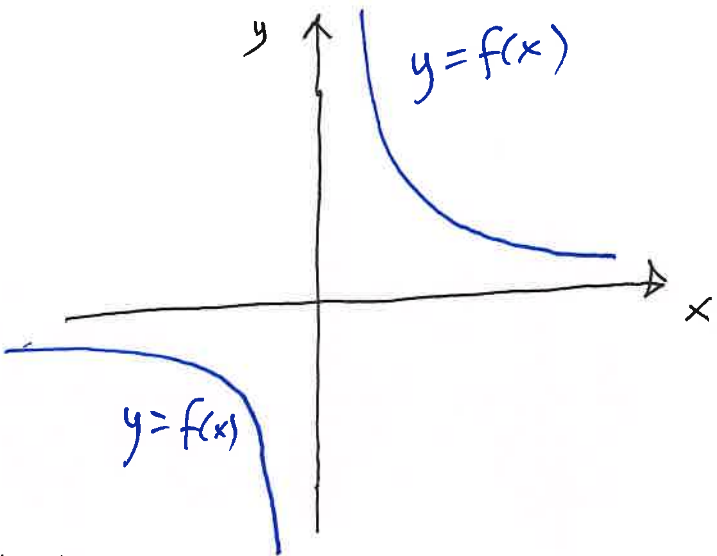
Conclusions: The line $x = 1$ is a vertical asymptote for $f(x)$.

The line $x = 5$ is also a vertical asymptote for $f(x)$.

Hyperbolas $\quad f(x) = \dfrac{1}{x}$

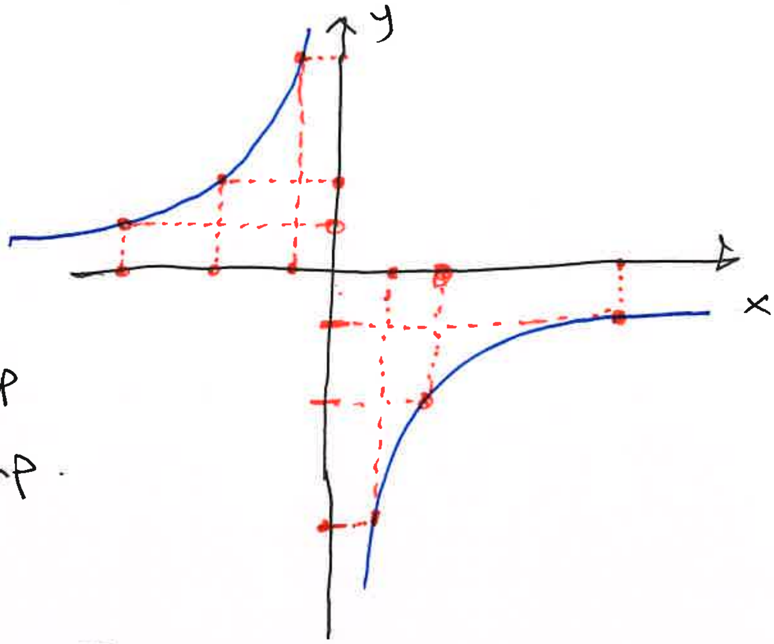
x	1	2	-1	-2	10	-10	0.1	-0.1	0.01
$f(x)$	1	0.5	-1	-0.5	0.1	-0.1	10	-10	100

Note: $f(x)$ is not defined for $x=0$



$x=0$ is a vertical asymptote and $y=0$ is a horizontal asymptote.

Ex: $f(x) = \frac{-1}{x}$



- not defined for $x = 0$

$x = 0$: vertical asymp

$y = 0$: horiz. asymp.

$$f(x) = \frac{-1}{x} \xrightarrow[x \to +\infty]{} 0^-$$

Ex: $f(x) = \frac{3x-5}{x-2}$ – also a hyperbola!

Polynomial division:

$$(3x-5) : (x-2) = 3 + \frac{1}{(x-2)} \qquad (x \neq 2)$$

$$\frac{-(3x-6)}{1}$$

(the remainder)

A hyperbola with vertical asymptote $x = 2$

and horizontal asymptote $y = 3$

Non-vertical asymptotes

Ex: $f(x) = x - 5 + \frac{2}{x-4} \qquad (x \neq 4)$

Note that $x = 4$ is a vertical asymptote

Put $g(x) = x - 5$

Then $f(x) - g(x) = \frac{2}{x-4} \xrightarrow[x \to \pm\infty]{} 0$

We say that $y = g(x)$ is a non-vertical asymptote for $f(x)$

Note: $f(x) = \dfrac{x^2 - 9x + 22}{x - 4}$ and polynomial

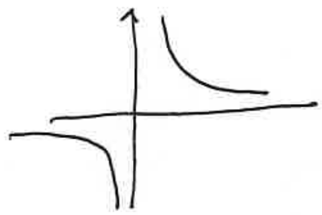
division gives good form $f(x) = x - 5 + \dfrac{2}{x-4}$

3. Continuity and the indermediate value thm.

A function is <u>continuous</u> if the graph is connected in all intervalls in the domain of definition.

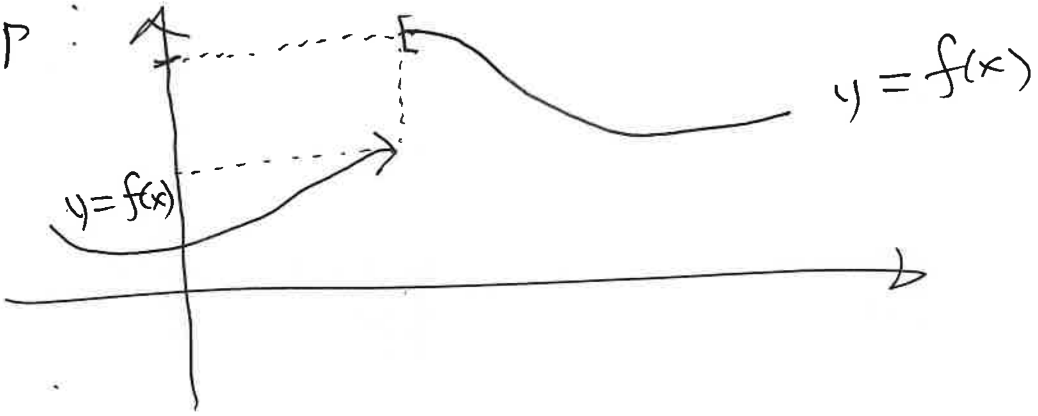
Ex: $f(x) = \frac{1}{x}$, $D_f = \langle \leftarrow, 0 \rangle \cup \langle 0, \rightarrow \rangle$

is continuous.



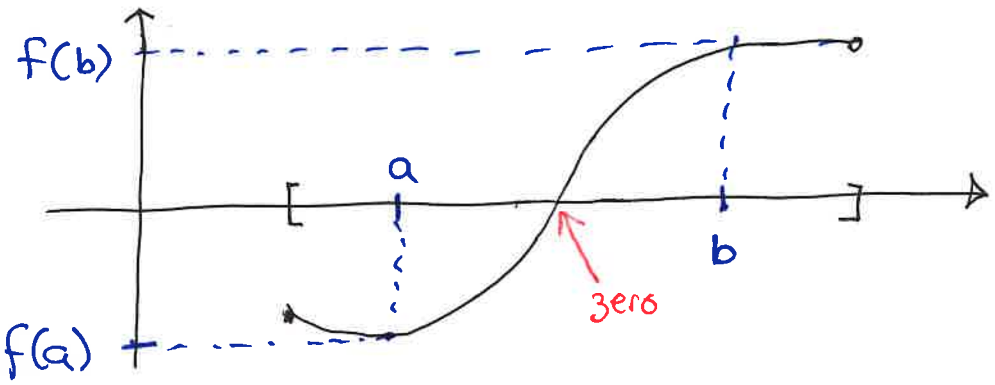
Fact: All 'usual' functions are continuous.

Not continuous: - then we have a jump:



The intermediate value theorem

If $f(x)$ is a continuous function in an interval I and $a, b \in I$ with $f(a) < 0$ and $f(b) > 0$ then there is a zero for $f(x)$ between a and b.



<u>Ex</u>: $f(x) = x\sqrt{2x+5} - \dfrac{10}{x}$ has a zero between $x=1$ and $x=10$ because

- $f(1) = 1 \cdot \sqrt{2 \cdot 1 + 5} - \dfrac{10}{1} = \sqrt{7} - 10 < 0$
- $f(10) = 10 \cdot \sqrt{2 \cdot 10 + 5} - \dfrac{10}{10} = 10 \cdot 5 - 1 > 0$
- $f(x)$ is continuous.

Then, by the IVT there is a zero between $x=1$ and $x=10$.