EBA2911 Mathematics for Business Analytics
autumn 2019
Exercises

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

R. Lucas

Lecture 10
Sec. 5.3, 4.9-10
Inverse functions. Exponential functions. Logarithms.

Here are recommended exercises from the textbook [SHSC].

Section **5.3** exercise 1, 3-5, 7, 9, 10
Section **4.9** exercise 1-4, 8-10
Section **4.10** exercise 1, 2, 6

Problems for the exercise session
Wednesday 16 Oct. from 14 o'clock in B2-085

Problem 1 Suppose $g(x)$ is the inverse function of $f(x)$. Determine:

a) $g(10)$ if $f(3) = 10$

b) $f(g(5))$

c) $f(\sqrt{2})$ if $g(3) = \sqrt{2}$

d) $g(f(9))$

Problem 2 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.

a) $f(x) = 2x - 3$ with D_f = all numbers

b) $f(x) = 0{,}5x + 1{,}5$ with D_f = all numbers

c) $f(x) = x^2 + 6x$ with $D_f = \langle \leftarrow, -3]$

d) $f(x) = 20 + \frac{1}{x-3}$ with $D_f = \langle 3, \rightarrow \rangle$

e) $f(x) = (x-1)^3 + 50$ with $D_f = [1, \rightarrow\rangle$

f)
$$f(x) = \begin{cases} \frac{10}{x} & \text{if } 0 < x \leqslant 10 \\ 2 - \frac{x}{10} & \text{if } 10 < x \leqslant 20 \end{cases}$$

Problem 3 Determine the expression $g(x)$ of the inverse function with domain and range.

a) $f(x) = \frac{4x-10}{x-3}$

b) $f(x) = \frac{70-40x}{3-2x}$

c) $f(x) = \sqrt{2x-3} + x$

Problem 4 We have (approximately) $\ln 2 = 0{,}6931$ and $\ln 3 = 1{,}0986$ and $\ln 5 = 1{,}6094$. Use these numbers to determine the values (approximately) without using the ln-button on the calculator.

a) $\ln 250$

b) $\ln 625$

c) $\ln \frac{625}{216}$

d) $\ln \frac{1\,000\,000}{27}$

e) $\ln 130 - \ln 78$

f) $\ln \sqrt[10]{6}$

Problem 5 Solve the equations.

a) $e^{2x+1} = 5$ b) $\ln(x-3) = -2$ c) $e^{2x+1} = 3e^{x+2}$

d) $\ln(x-3) = \ln(2x+1)+1$ e) $e^{2x} - 4e^{x} - 5 = 0$ f) $\frac{20\ln\sqrt{x}}{1-\ln x} = 10$

Problem 6 Solve the inequalities.

a) $e^{2x+1} \geqslant 5$ b) $\ln(x-3) < -2$ c) $\frac{3e^{x}}{e^{x}+1} < 5$ d) $\ln\frac{3x-2}{x-7} \geqslant 0$

Problem 7 Determine the asymptotes of the function.

a) $f(x) = e^{-0.1x} + 23$ b) $f(x) = \ln(10-x)$

c) $f(x) = e^{x(10-x)} + 50$ d) $f(x) = (100x^3 + 70x + 1000)e^{-\frac{x}{100}}$

e) $f(x) = \frac{100e^{0,04x}}{e^{0,04x}+50}$ f) $f(x) = \ln(x^2 - 400)$

g) $f(x) = \ln(120x+10) - \ln(20x-30)$, $D_f = \langle \frac{3}{2}, \rightarrow\rangle$

Problem 8 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.
(a) $f(x) = e^{\frac{x}{3}} - 1$ with $D_f = [0, \rightarrow\rangle$ (b) $f(x) = 4\ln(x-10)$ with $D_f = [11, \rightarrow\rangle$
(c) $f(x) = e^{\frac{2}{x+10}}$ with $D_f = [0, \rightarrow\rangle$ (d) $f(x) = \ln(x^2 - 6x + 7)$ with $D_f = [0, 1\rangle$

Problem 9 Determine the expression $f(x) = c + \frac{a}{x-b}$ of the hyperbolas (a-d) in figure 1.

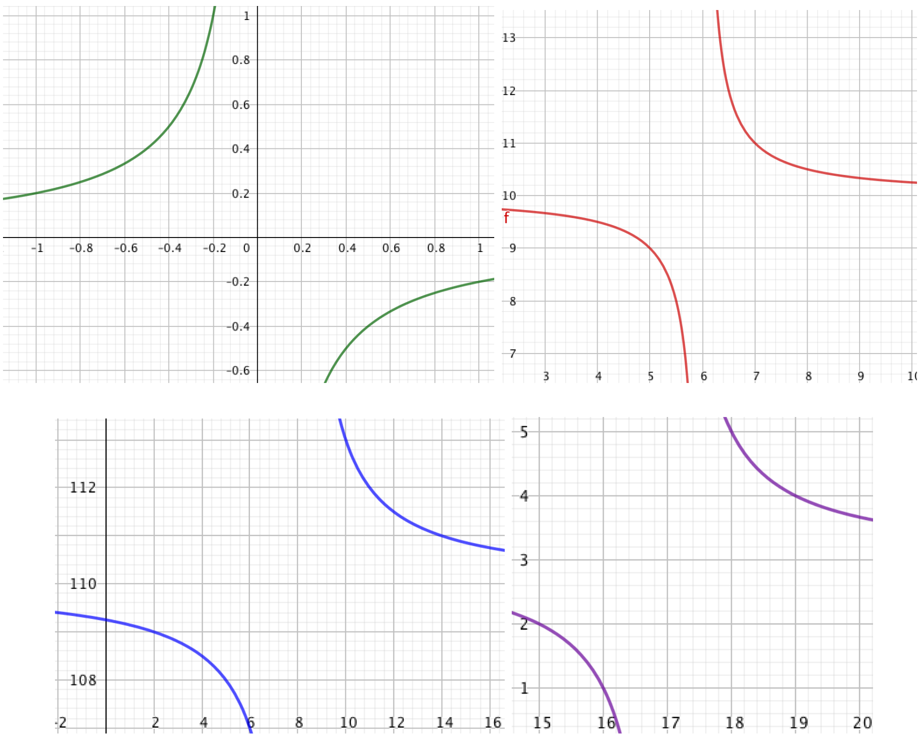

Figure 1: Hyperbolas a-d

Problem 10 Determine the asymptotes of the hyperbolas (a-d) in Problem 9.
Problem 11 Determine the asymptotes of the rational functions.

a) $f(x) = \frac{4x-10}{x-3}$ b) $f(x) = \frac{70-40x}{3-2x}$ c) $f(x) = \frac{3x^2-6x+8}{x^2+3}$

d) $f(x) = \frac{4x^2-28x+40}{x^2-4x+3}$ e) $f(x) = \frac{x^2+3x+5}{x-7}$ f) $f(x) = \frac{x^3-8}{x^2-10x+16}$

Problem 12 Determine if the function $f(x)$ has a zero in the interval I. Hint: The intermediate value theorem!
a) $f(x) = \sqrt{x-2} - x + 3$ and $I = [4, 5]$
b) $f(x) = (x-5)\sqrt{(0{,}2x+5)} - 0{,}2(x-3)^2$ and $I = [5, 15]$
c) $f(x) = \frac{4x-10}{x-3} - 4$ and $I = [2, 4]$

Answers

Problem 1

a) 3   b) 5   c) 3   d) 9

Problem 2

a) $g(x) = 0{,}5x + 1{,}5$ with D_g = all numbers
b) $g(x) = 2x - 3$, D_g = all numbers
c) $g(x) = -3 - \sqrt{x+9}$, $D_g = V_f = [-9, \rightarrow\rangle$
d) $g(x) = 3 + \frac{1}{x-20}$, $D_g = \langle 20, \rightarrow\rangle$
e) $g(x) = \sqrt[3]{x-50} + 1$, $D_g = [50, \rightarrow\rangle$
f)

$$g(x) = \begin{cases} \frac{10}{x} & \text{if } x \geqslant 1 \\ 20 - 10x & \text{if } 0 \leqslant x < 1 \end{cases}$$

Oppgave 3

a) $g(x) = \frac{3x-10}{x-4}$, $D_g = \langle \leftarrow, 4\rangle \cup \langle 4, \rightarrow\rangle$ which are all numbers except 4, $V_g = \langle \leftarrow, 3\rangle \cup \langle 3, \rightarrow\rangle$ which are all numbers except 3.
b) $g(x) = \frac{70-3x}{40-2x}$, $D_g = \langle \leftarrow, 20\rangle \cup \langle 20, \rightarrow\rangle$, $V_g \langle \leftarrow, \frac{3}{2}\rangle \cup \langle \frac{3}{2}, \rightarrow\rangle$
c) $g(x) = x + 1 - \sqrt{2x-2}$, $D_g = [\frac{3}{2}, \rightarrow\rangle$ (even if the expression is defined for $x \geqslant 1$), $V_g = [\frac{3}{2}, \rightarrow\rangle$.

Problem 4

a) $\ln 250 = \ln 2 + 3\ln 5 = 0{,}6931 + 3 \cdot 1{,}6094 = 5{,}5213$
b) $\ln 625 = 4\ln 5 = 4 \cdot 1{,}6094 = 6{,}4376$
c) $\ln \frac{625}{216} = 4\ln 5 - 3(\ln 3 + \ln 2) = 4 \cdot 1{,}6094 - 3(1{,}0986 + 0{,}6931) = 1{,}0625$
d) $\ln \frac{1\,000\,000}{27} = 6(\ln 5 + \ln 2) - 3\ln 3 = 6 \cdot (1{,}6094 + 0{,}6931) - 3 \cdot 1{,}0986 = 10{,}5192$
e) $\ln 130 - \ln 78 = \ln 5 + \ln 26 - \ln 3 - \ln 26 = 1{,}6094 - 1{,}0986 = 0{,}5108$
f) $\ln 6^{\frac{1}{10}} = \frac{1}{10} \cdot \ln 6 = \frac{1{,}0986 + 0{,}6931}{10} = 0{,}1792$

Problem 5

a) $x = \frac{1}{2}(\ln(5) - 1)$
b) $x = 3 + e^{-2}$
c) $x = 1 + \ln(3)$
d) $x = -\dfrac{e+3}{2e-1}$
e) $x = \ln 5$
f) $x = e^{0{,}5}$

Problem 6

a) Because $\ln x$ is a strictly increasing function for $x > 0$ we can insert the left hand side and the right hand side into $\ln x$ and keep the inequality. It gives $x \geqslant \frac{1}{2}(\ln 5 - 1)$.
b) Because e^x is a strictly increasing function we can insert the left hand side and the right hand side into e^x and keep the inequality. It gives $3 < x < 3 + e^{-2}$.
c) All numbers on the number line (are called the real numbers and written as $\mathbb{R}$, i.e. $x \in \mathbb{R}$).
d) Note that the inequality only is defined for $x < \frac{3}{2}$ and for $x > 7$. We insert the left and right hand side into e^x and keep the inequality. This gives $\frac{3x-2}{x-7} \geqslant 1$ which we then solve: $x \leqslant -\frac{5}{2}$ or $x > 7$ (and this is within the domain of definition of the inequality). Alternate way of writing: $x \in \langle \leftarrow, -\frac{5}{2}] \cup \langle 7, \rightarrow\rangle$.

Problem 7

a) horizontal asymptote: $y = 23$

b) vertical asymptote: $x = 10$

c) horizontal asymptote: $y = 50$

d) horizontal asymptote: $y = 0$

e) horizontale asymptotes: $y = 100$ ($x \to \infty$) and $y = 0$ ($x \to -\infty$)

f) vertical asymptotes: $x = \pm 20$

g) vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = \ln 6$

Problem 8

a) $g(x) = 3\ln(x+1)$, $D_g = V_f = [0, \rightarrow\rangle$

b) $g(x) = e^{\frac{x}{4}} + 10$, $D_g = [0, \rightarrow\rangle$

c) $g(x) = \frac{2}{\ln x} - 10$, $D_g = \langle 1, \sqrt[5]{e}]$

d) $g(x) = 3 - \sqrt{e^x + 2}$, $D_g = \langle \ln 2, \ln 7]$

Problem 9

a) $f(x) = -\frac{1}{5x}$ b) $f(x) = 10 + \frac{1}{x-6}$ c) $f(x) = 110 + \frac{6}{x-8}$ d) $f(x) = 3 + \frac{2}{x-17}$

Problem 10

a) vertical asymptote: $x = 0$, horizontal asymptote: $y = 0$
b) vertical asymptote: $x = 6$, horizontal asymptote: $y = 10$
c) vertical asymptote: $x = 8$, horizontal asymptote: $y = 110$
d) vertical asymptote: $x = 17$, horizontal asymptote: $y = 3$

Problem 11

a) $f(x) = 4 + \frac{2}{x-3}$ so vertical asymptote: $x = 3$, horizontal asymptote: $y = 4$
b) $f(x) = 20 + \frac{10}{3-2x}$ so vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = 20$
c) $f(x) = 3 - \frac{6x+1}{x^2+3}$ so no vertical asymptote, horizontal asymptote: $y = 3$
d) $f(x) = 4 - \frac{4(3x-7)}{(x-1)(x-3)}$ so vertical asymptotes: $y = 1$ and $y = 3$, horizontal asymptote: $x = 4$
e) $f(x) = x + 10 + \frac{75}{x-7}$ so vertical asymptote: $x = 7$, non-vertical asymptote: $y = x + 10$
f) $f(x) = x + 10 + \frac{84}{x-8}$ so vertical asymptote: $x = 8$, non-vertical asymptote: $y = x + 10$

Problem 12

a) $f(x)$ has a zero between $x = 4$ and $x = 5$ by the intermediate value theorem because $f(4) = \sqrt{4-2} - 4 + 3 = 0{,}41 > 0$ while $f(5) = \sqrt{5-2} - 5 + 3 = -0{,}27 < 0$ and the function is defined and is continuous on the whole interval.

b) $f(x)$ has a zero between $x = 5$ and $x = 6$ by the intermediate value theorem because $f(5) = -0{,}80$ while $f(6) = 0{,}69 > 0$ and the function is defined and is continuous on the whole interval.
Note: $f(15) = -0{,}52 < 0$ together with $f(6) > 0$ tell that $f(x)$ has a zero between $x = 6$ and $x = 15$. So f(x) has at leas 2 zeros on the interval $[5, 15]$.

c) $f(x) = \frac{2}{x-3}$ har no zeros on the interval $I = [2, 4]$ because the equation $\frac{2}{x-3} = 0$ has no solutions. Note: We can not use the intermediate value theorem even if $f(2) = -2 < 0$ and $f(4) = 2 > 0$ because $f(x)$ is not defined on the whole interval (even if $f(x)$ er continuous for all x where it is defined).