EBA2911 Mathematics for Business Analytics
autumn 2019
Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

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Lecture 15
Sec. 7.4-6
Linearisation. Taylor polynomials.

Here are recommended exercises from the textbook [SHSC].

Section **7.4** exercise 1-3
Section **7.5** exercise 1-3

Problems for the exercise session
Wednesday 20 Nov. from 14 o'clock in B2-085

Problem 1
a) Determine the Taylor polynomials $P_1(x), \dots, P_4(x)$ of degree $1-4$ of the function $f(x) = e^x$ at 0.
b) Compute $P_1(1), \dots, P_4(1)$ and compute how good approximations these values give to $f(1) = e$.

Problem 2
a) Determine the Taylor polynomials $P_1(x), \dots, P_4(x)$ of degree $1-4$ of the function $f(x) = xe^x$ at 0.
b) Compute $P_1(1), \dots, P_4(1)$ and compute how good approximations these values give to $f(1) = e$.

Problem 3
a) Determine the Taylor polynomials $P_1(x), \dots, P_4(x)$ of degree $1-4$ of the function $f(x) = \ln(x)$ at 1.
b) Compute $P_1(2), \dots, P_4(2)$ and compute how good approximations these values give to $f(2) = \ln(2)$.

Problem 4 Determine the Taylor polynomials $P_1(x), \dots, P_4(x)$ of degree $1-4$ of the function $f(x) = x^4$ at 0.

Problem 5 Determine the Taylor polynomials $P_1(x), \dots, P_4(x)$ of degree $1-4$ of the function $f(x) = \frac{1}{1-x}$ at 0.

Problem 6 Let $P_1(x), \dots, P_4(x)$ be the Taylor polynomials in problem 1. Compute the limit values.

$$\text{a)} \lim_{x\to 0} \frac{e^x - P_1(x)}{x^2} \quad \text{b)} \lim_{x\to 0} \frac{e^x - P_2(x)}{x^3} \quad \text{c)} \lim_{x\to 0} \frac{e^x - P_3(x)}{x^4} \quad \text{d)} \lim_{x\to 0} \frac{e^x - P_4(x)}{x^5}$$

Problem 7 Let $P_1(x), \dots, P_4(x)$ be the Taylor polynomials in problem 3. Compute the limit values.

$$\text{a)} \lim_{x\to 1} \frac{\ln(x) - P_1(x)}{(x-1)^2} \quad \text{b)} \lim_{x\to 1} \frac{\ln(x) - P_2(x)}{(x-1)^3} \quad \text{c)} \lim_{x\to 1} \frac{\ln(x) - P_3(x)}{(x-1)^4} \quad \text{d)} \lim_{x\to 1} \frac{\ln(x) - P_4(x)}{(x-1)^5}$$

Problem 8 (Multiple choice exam 2017s, problem 12)
We consider the price elasticity $\varepsilon = \varepsilon(p)$ of a commodity with demand function $D(p) = 120 - 8p$. Then:

(A) $\varepsilon > -1$ for $p = 7{,}5$
(B) $\varepsilon > -1$ for $p < 7{,}5$
(C) $\varepsilon > -1$ for $p > 7{,}5$
(D) $\varepsilon > -1$ for all values of p
(E) I choose not to solve this problem.

Problem 9 (Multiple choice exam 2016a, problem 12)
Demand for a commodity is given as $D(p) = 110 - 5p$. Then the elasticity $\varepsilon(p) = -1$ for:

(A) $p = 7$
(B) $p = 11$
(C) $p = \frac{16}{5}$
(D) $p = 22$
(E) I choose not to solve this problem.

Problem 10 (Multiple choice exam 2017s, problem 4)
A firm has the cost function $C(x) = 205x^3 - 120x^2 + 2000x + 2800$ when $x \geqslant 0$. What is the minimal average unit cost (the cost optimum)?

(A) 2 kr
(B) 12 kr
(C) 3980 kr
(D) 7960 kr
(E) I choose not to solve this problem.

Problem 11 (Multiple choice exam 2016a, problem 14)
We consider the limit value

$$\lim_{x\to\infty} \frac{1 - x\ln(x)}{e^x}$$

What is true?

(a) The limit value does not exist
(b) The limit value equals 1
(c) The limit value equals $-\frac{1}{2}$
(d) The limit value equals 0
(e) I choose not to solve this problem.

Problem 12 (Multiple choice exam 2015a, problem 15)
We consider the limit value

$$\lim_{x\to 1} \frac{\ln(x) - x + 1}{x^2 - 2x + 1}$$

What is true?

(a) The limit value does not exist
(b) The limit value equals 0
(c) The limit value equals 1
(d) The limit value equals $-\frac{1}{2}$
(e) I choose not to solve this problem.

Fasit

Problem 1

a) $P_1(x) = 1 + x$, $P_2(x) = 1 + x + \frac{x^2}{2}$, $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

b) $P_1(1) = 2$, $P_2(1) = 2{,}5$, $P_3(1) = \frac{8}{3} \approx 2{,}67$, $P_4(x) = \frac{65}{24} \approx 2{,}71$. The distance from $f(1) = e$ equals (approximately): $|f(1) - P_1(1)| = |e - 2| = 0{,}72$, $|f(1) - P_2(1)| = |e - 2{,}5| = 0{,}22$, $|f(1) - P_3(1)| = |e - \frac{8}{3}| = 0{,}052$, $|f(1) - P_4(1)| = |e - \frac{65}{24}| = 0{,}0099$

Problem 2

a) $P_1(x) = x$, $P_2(x) = x + x^2$, $P_3(x) = x + x^2 + \frac{x^3}{2}$, $P_4(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$

b) $P_1(1) = 1$, $P_2(1) = 2$, $P_3(1) = 2{,}5$, $P_4(x) = \frac{8}{3} \approx 2{,}67$. The distance from $f(1) = e$ equals (approximately): $|f(1) - P_1(1)| = |e - 1| = 1{,}72$, $|f(1) - P_2(1)| = |e - 2| = 0{,}72$, $|f(1) - P_3(1)| = |e - 2{,}5| = 0{,}22$, $|f(1) - P_4(1)| = |e - \frac{8}{3}| = 0{,}052$

Problem 3

a) $P_1(x) = (x-1)$, $P_2(x) = (x-1) - \frac{(x-1)^2}{2}$, $P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$, $P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$

b) $P_1(2) = 1$, $P_2(2) = \frac{1}{2}$, $P_3(1) = \frac{5}{6} \approx 0{,}83$, $P_4(x) = \frac{7}{12} \approx 0.58$. The distance from $f(2) = \ln(2)$ equals (approximately): $|f(2) - P_1(2)| = |\ln(2) - 1| = 0{,}31$, $|f(2) - P_2(2)| = |\ln(2) - \frac{1}{2}| = 0{,}19$, $|f(2) - P_3(2)| = |\ln(2) - \frac{5}{6}| = 0{,}14$, $|f(2) - P_4(2)| = |\ln(2) - \frac{7}{12}| = 0{,}11$

Problem 4 $P_1(x) = 0$, $P_2(x) = 0$, $P_3(x) = 0$, $P_4(x) = x^4$

Problem 5 $P_1(x) = 1 + x$, $P_2(x) = 1 + x + x^2$, $P_3(x) = 1 + x + x^2 + x^3$, $P_4(x) = 1 + x + x^2 + x^3 + x^4$

Problem 6

a) This is a $\frac{0}{0}$-expression. Hence we can use l'Hôpital's rule. Differentiate numerator and denominator. Get another $\frac{0}{0}$-expression and use l'Hôpital's rule again:

$$\lim_{x\to 0} \frac{e^x - (1+x)}{x^2} \overset{\text{l'Hôp}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \overset{\text{l'Hôp}}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

b)

$$\lim_{x\to 0} \frac{e^x - (1 + x + \frac{x^2}{2})}{x^3} \overset{\text{l'Hôp}}{=} \lim_{x\to 0} \frac{e^x - (1+x)}{3x^2} \overset{\text{l'Hôp}}{=} \lim_{x\to 0} \frac{e^x - 1}{6x} \overset{\text{l'Hôp}}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$$

c) $\frac{1}{24}$

d) $\frac{1}{120}$

Problem 7

a) This is a $\frac{0}{0}$-expression. Hence we can use l'Hôpital's rule. Differentiate numerator and denominator. Get another $\frac{0}{0}$-expression and use l'Hôpital's rule again:

$$\lim_{x\to 1} \frac{\ln(x) - (x-1)}{(x-1)^2} \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{\frac{1}{x} - 1}{2(x-1)} \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}$$

b)

$$\lim_{x\to 1} \frac{\ln(x) - [(x-1) - \frac{(x-1)^2}{2}]}{(x-1)^3} \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{\frac{1}{x} - [1 - (x-1)]}{3(x-1)^2} \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{-\frac{1}{x^2} + 1}{6(x-1)} \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{\frac{2}{x^3}}{6} = \frac{1}{3}$$

c) $-\frac{1}{4}$

d) $\frac{1}{5}$

Problem 8 (Multiple choice exam 2017s, problem 12)
B

Problem 9 (Multiple choice exam 2016a, problem 12)
B

Problem 10 (Multiple choice exam 2017s, problem 4)
C

Problem 11 (Multiple choice exam 2016a, problem 14)
D

Problem 12 (Multiple choice exam 2015a, problem 15)
D